Solving the biharmonic equation with conforming finite elements with Comsol Multiphysics

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The biharmonic equation

\[ \Delta^2 u = f \quad \text{in } \Omega, \]
\[ u = g, \quad \Delta u = h \quad \text{on } \partial \Omega. \]

\[ \Delta^2 u = \Delta(\Delta u) = \sum_{i,j=1}^{n} \frac{\partial^4 u}{\partial^2 x_i \partial^2 x_j} \]

The variational formulation is

\[ (\Delta u, \Delta v) = (f, v) + \left\langle h, \frac{\partial v}{\partial n} \right\rangle_{\partial \Omega} \quad \forall v \in V := H^2(\Omega) \cap H^1_0(\Omega), \]
Step by step instructions

- In this example, we take

\[ \Omega = (0, 1)^2, \quad f = 4\pi^4 \sin(\pi x) \sin(\pi y), \quad g = 0, \quad h = 0, \]

so that the exact solution is \( u = \sin(\pi x) \sin(\pi y) \).
Step by step instructions

Step 1: Start the application Comsol Multiphysics. The path to the executable is

/usr/local/packages/comsol35a/bin/comsol
Step by step instructions

Step 2:  (a) Select ‘PDE Modes→Weak Form, Subdomain→ Stationary analysis’
(b) Choose ‘Argyris-Quintic’ in the ‘Element’ pull-down tab
(c) Select ‘OK’
Step 3:  
(a) Select ‘Draw→Specific Object→Square’  
(b) Select ‘OK’
Step by step instructions

Step 4: (optional) Click the icon for “Zoom Extents” - it is the icon with a magnifying glass and a red cross
Step by step instructions

Step 5:  (a) Select ‘Options→Global Expressions’
(b) Enter the following information in the given box:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>$4\pi^4\sin(\pi x)\sin(\pi y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>hh</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>exactsoln</td>
<td>$\sin(\pi x)\sin(\pi y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>error</td>
<td>$\text{abs}(u-\text{exactsoln})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>l2err</td>
<td>$\text{error}^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Select ‘Ok’

(The variable $h$ is already pre-defined in Comsol, so we cannot use it as a global expression)
Step by step instructions

Step 6:  
(a) Select ‘Physics→Subdomain Settings’
(b) In the ‘Weak’ tab, enter the following information in each field:

\[
\text{weak} : \quad (\text{uxx} + \text{uyy}) \ast (\text{test}(\text{uxx}) + \text{test}(\text{uyy})) - f \ast \text{test}(u)
\]

\[
\text{dweak} : \quad 0
\]

\[
\text{bnd.weak} : \quad 0
\]

\[
\text{constr} : \quad 0
\]

Constraint type : ideal

\[
\text{constrf} : \quad 0
\]

(d) Select the ‘Element’ tab
(e) Change the Constraint order (cporder) to 2
(f) Select ‘OK’
Step by step instructions

Step 6:

Subdomain Settings - Weak Form, Subdomain (w)

- **Weak terms**
  - weak
  - dweak
  - bound weak
  - constr
  - Constraint type: ideal
  - Constraint force

- **Element settings**
  - Shape functions
  - Integration order
  - Constraint order
  - Integration order for ultraweak term
Step by step instructions

Step 7:  
(a) Select ‘Physics→Boundary Settings’

(b) Under the “Weak” tab, select all four boundaries (1,2,3,4) and enter the following information:

\[
\begin{align*}
\text{weak} & : \quad -hh \times (\text{test(ux)} \times nx + \text{test(uy)} \times ny) \\
\text{dweak} & : \quad 0 \\
\text{constr} & : \quad -u + g \\
\text{Constraint type} & : \quad \text{ideal} \\
\text{constrf} & : \quad 0
\end{align*}
\]

(c) Select ‘OK’
Step by step instructions

Step 7:
Step by step instructions

Step 8: Select the ‘Solve’ icon (the icon with a plain equal sign)