Introduction to Numerical Relativity II

Erik Schnetter, Pohang, July 2007

Lectures Overview

I. The Einstein Equations
(Formulations and Gauge Conditions)

II. Analysis Methods
(Horizons and Gravitational Waves)

III. Numerical methods
(Cactus and Mesh Refinement)

Analysis methods: Horizons and Waves

1. Event horizons
2. Apparent, isolated, and dynamical horizons
3. Gravitational waves

Please interrupt and ask questions at any time

Part 1:
Event Horizons

Why look at event horizons?

- Event horizons tell us about black holes, their location, size, and shape
- Event horizons are gauge-independent (coordinate-independent)
- They also tell us what can be seen from infinity

Event horizon

- In some regions of spacetime curvature is so strong that nothing can get away to infinity
- The boundary of these regions is called *event horizon*
- By definition, things inside an event horizon are not visible to an outside observer

Peculiarities

- The spacetime near an event horizon is not special. Event horizons can even exist in flat regions of spacetime
- As defined, event horizons move with the speed of light
- There could be an event horizon passing through this room right now; we would not notice

River model

- Imagine a river, which is slow upstream and has a waterfall downstream
- Imagine a boat on the river
- Near the waterfall the river flows faster than the boat moves
- Now consider that the speed of the river changes with time
- Where is the "point of no return"?

Details

- Event horizons are null surfaces (almost everywhere)
- Event horizons are *acausal*: When and where an event horizon is located depends on what is going to happen in the future
- Therefore, finding event horizons requires the whole future of the spacetime; it cannot be done locally in time
- (Analytic solution: Vaidya metric)

Algorithm

- See e.g. [Diener, Class. Quantum Grav. **20**, 4901 (2003)]
- Event horizons are found backwards in time:
  - one starts with a surface close to the EH at some late time,
  - and then tracks null geodesics backwards in time

A new general purpose event horizon finder for 3D numerical spacetimes

Peter Diener

Abstract

I present a new general purpose event horizon finder for full 3D numerical spacetimes. It works by evolving a complete null surface backwards in time. The null surface is described as the zero-level set of a scalar function, which in principle is defined everywhere. This description of the surface allows the surface, trivially, to change topology, making this event horizon finder able to handle numerical spacetimes where two (or more) black holes merge into a single final black hole.

Level set surfaces

$f(t, x^i) = 0,$ implicit horizon surface

better than explicit representation

$$f(t, r, \theta) = r - s(t, \theta),$$

$g^{\alpha\beta} \partial_\alpha f \partial_\beta f = 0.$ null condition

Evolution equation:

$$\partial_t f = \beta^i \partial_i f - \sqrt{\alpha^2 \gamma^{ij} \partial_i f \partial_j f},$$

Re-initialization

- The gradient of f becomes steeper with time near f=0, leading to problems
- Therefore f must be re-initialized from time to time
- Re-initialization changes the shape of f, but does not change where f is zero
- Pseudo-evolution: $\frac{\mathrm{d}f}{\mathrm{d}\lambda} = -\frac{f}{\sqrt{f^2+1}}\left(|\nabla f| - 1\right)$

Remarks

- Start with the apparent horizon at a late, stationary time, then evolve backwards
- The event horizon is an attractor (when evolving backwards in time)
- Therefore the solution becomes more accurate with time, even if the starting surface is not accurate

Example calculation

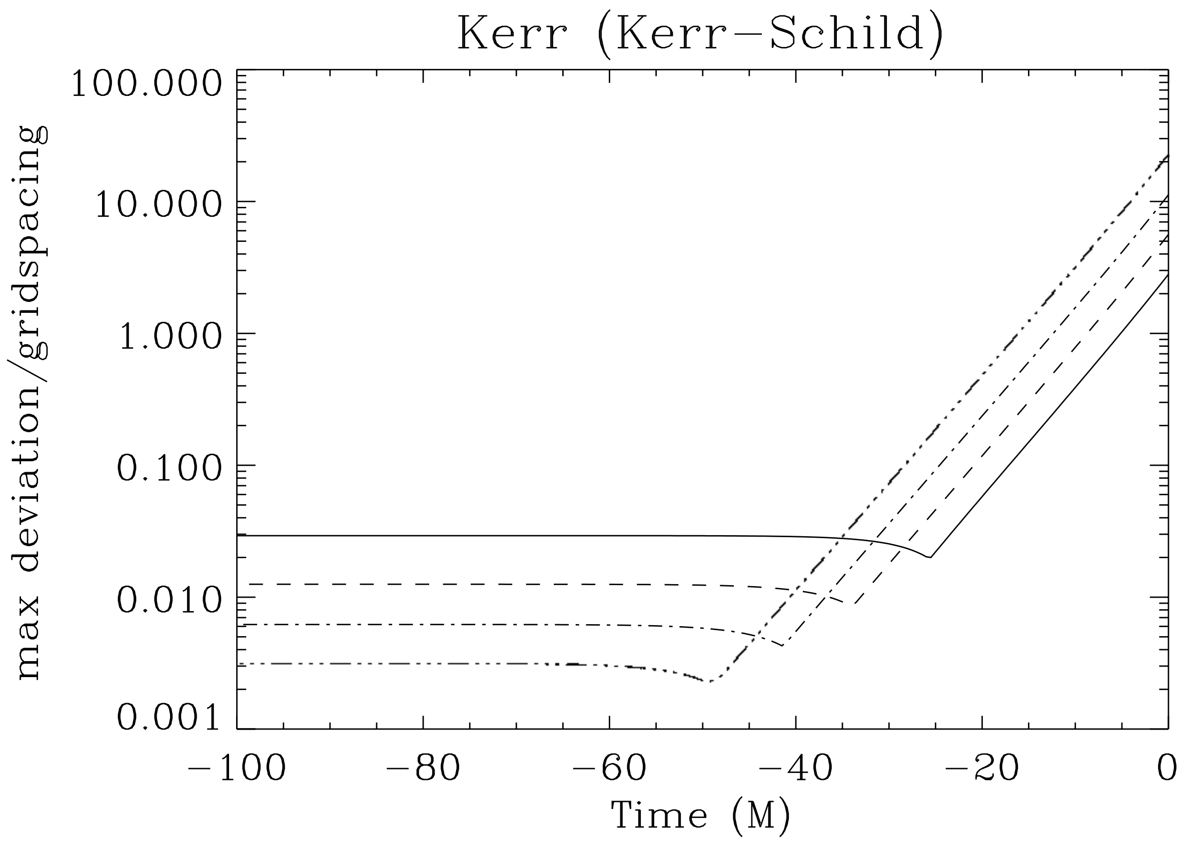


Figure 3. The maximum deviation, as defined in equation (14), divided by the grid spacing for a Kerr black hole of mass $M = 1$ and $a = 0.8$ in Kerr–Schild coordinates as function of time. The solid line is $\Delta = 0.2M$, the dashed line is $\Delta = 0.1M$, the dashed-dotted line is $\Delta = 0.05M$ and the dashed-triple-dot line is $\Delta = 0.025M$.

Horizon mass

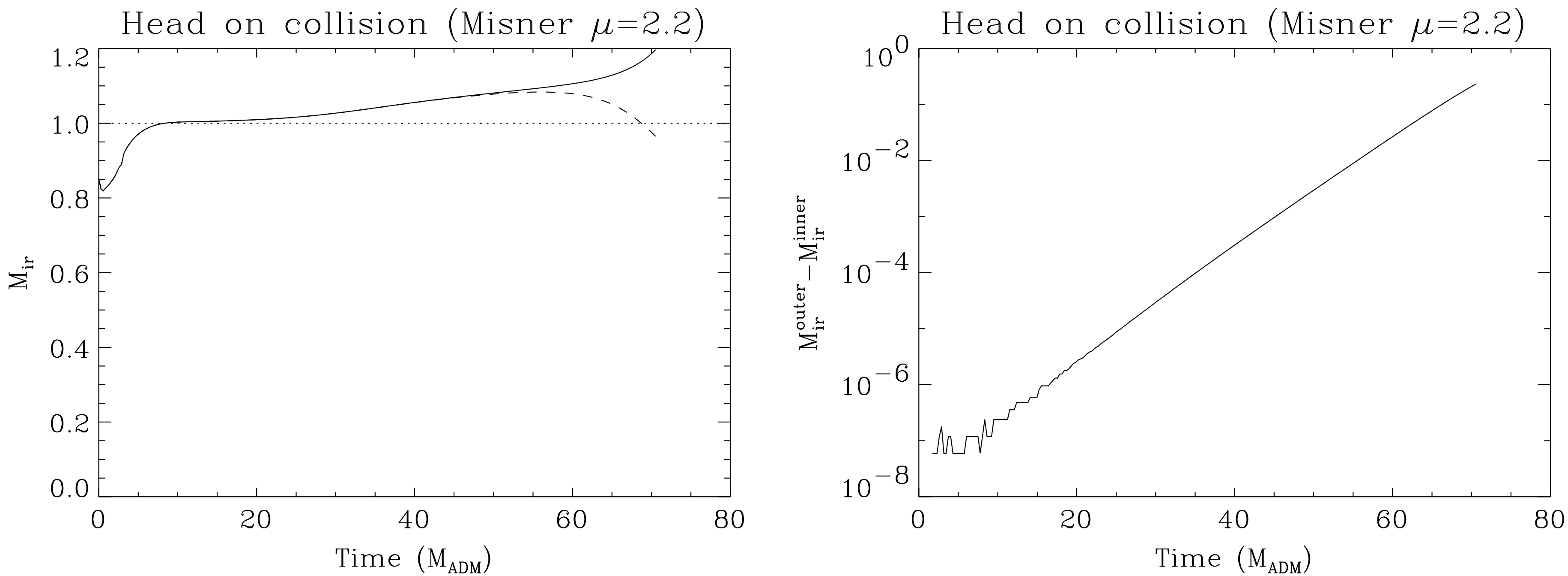
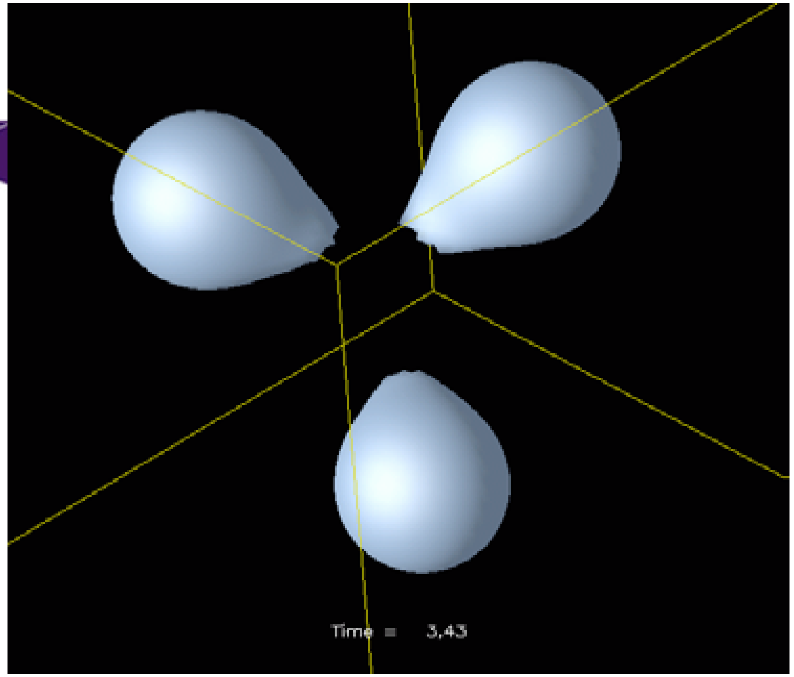
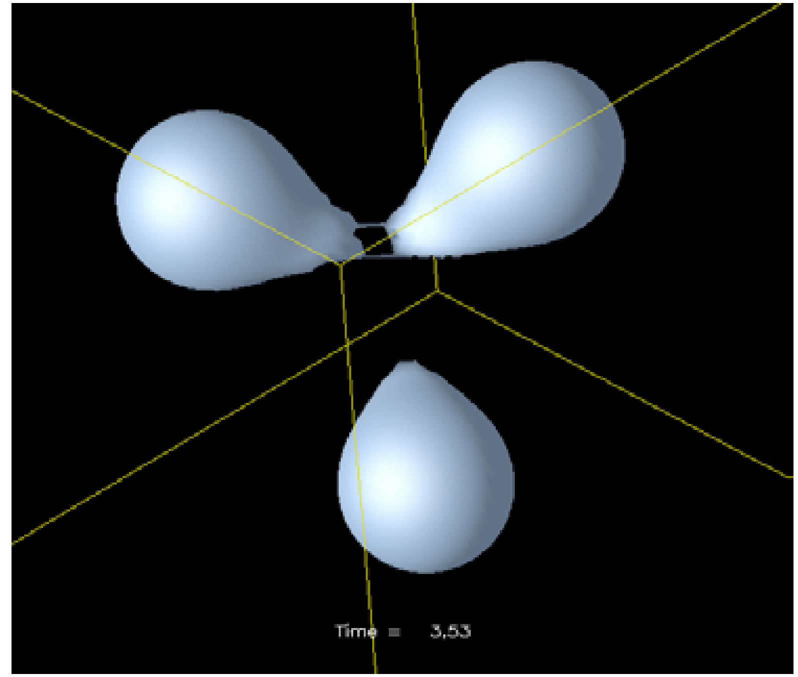


Figure 4. The plot on the left shows the total irreducible mass of the horizons in the Misner $\mu = 2.2$ spacetime as a function of time. Curves are shown for two different choices of surfaces at $T = 70M$. The solid line is for a sphere with radius $r_O = 3.0M$ and the dashed line is for a sphere with radius $r_I = 2.5M$. The horizontal dotted line denotes the ADM mass of the spacetime. The plot on the right shows the difference in mass of the two surfaces.

irreducible mass (ignoring spin): $M_{\rm ir} = \sqrt{\frac{\sum_{i=1}^{n} A_i}{16\pi}},$



(a)



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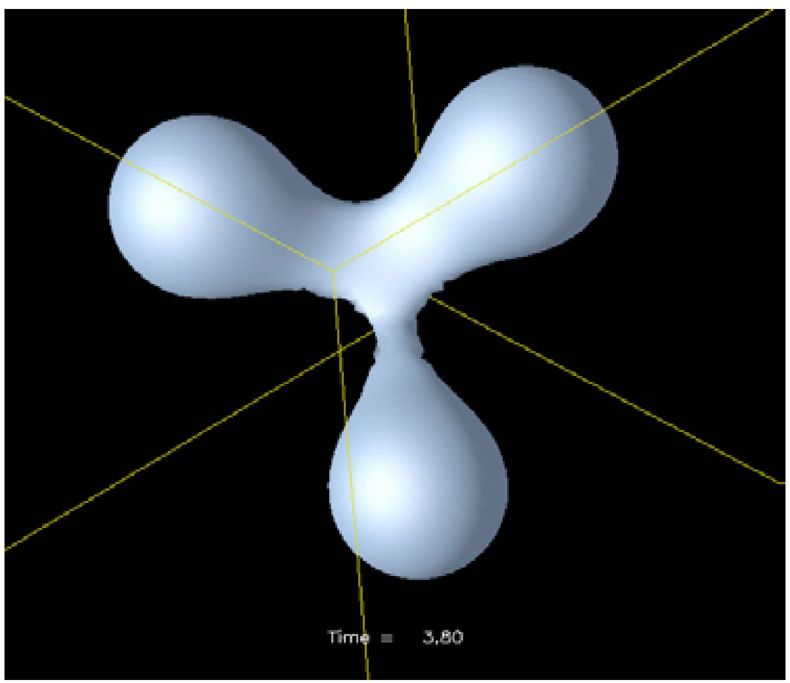

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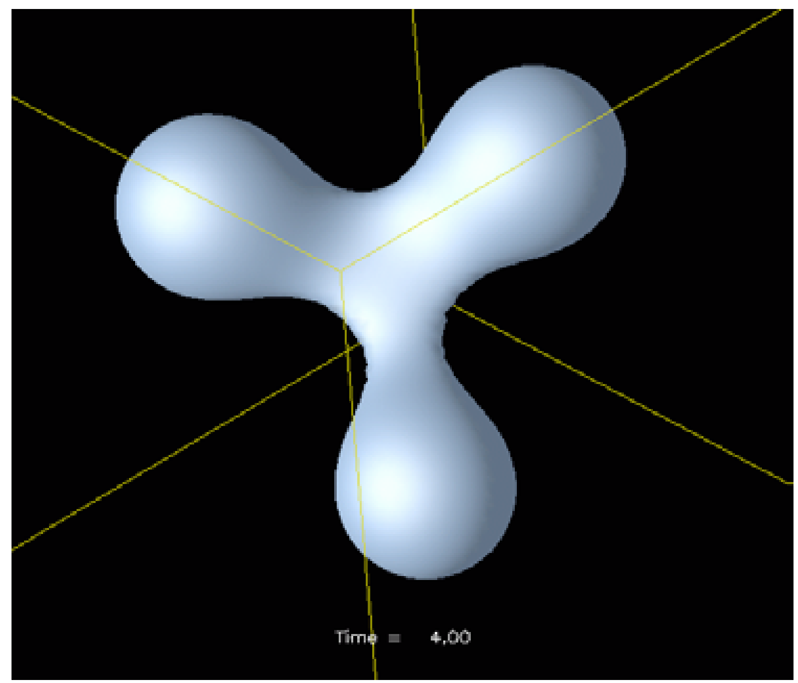
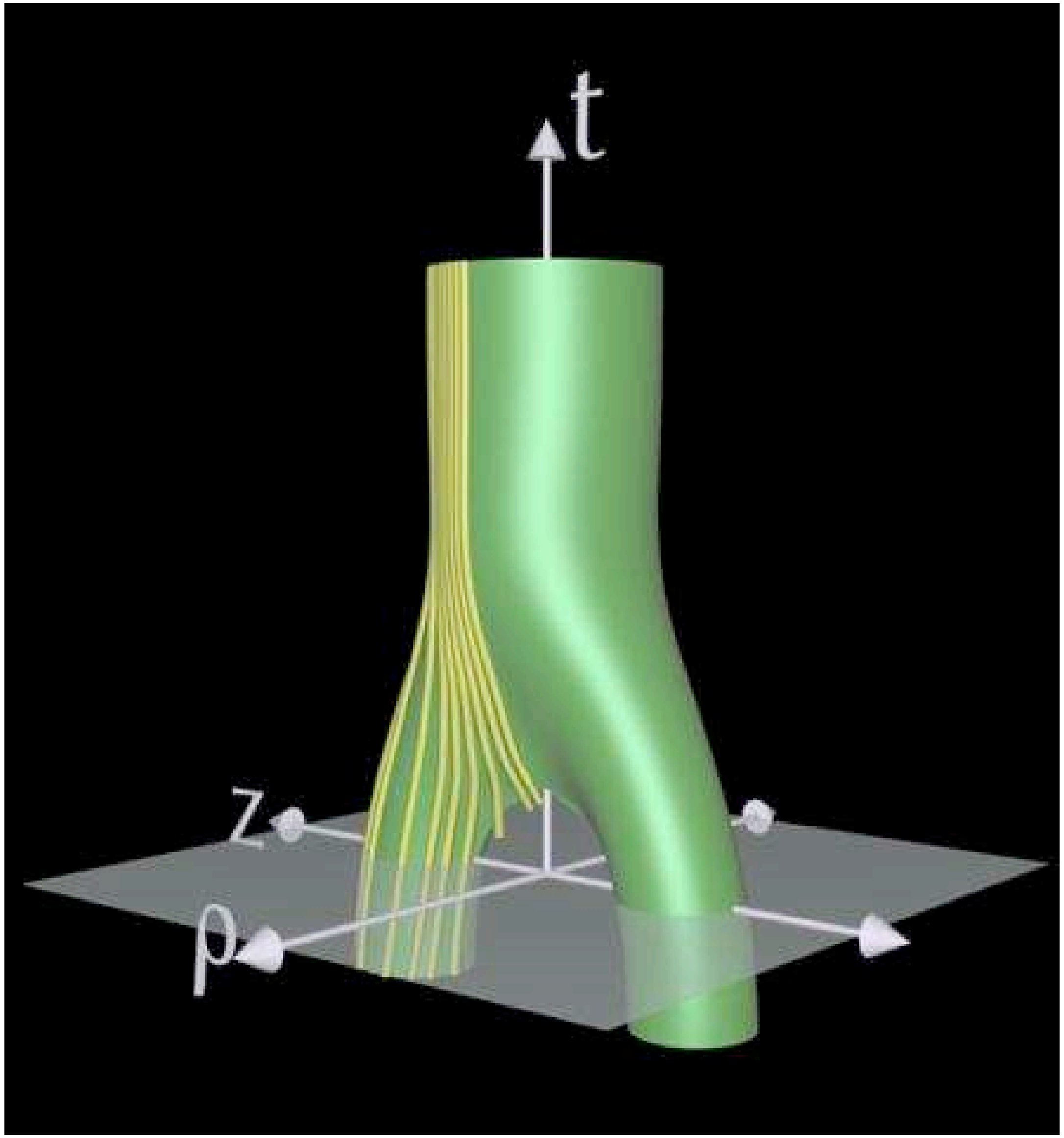


(d)

Pair of pants



Matzner et al., Science **270**, 941 (1995)

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Part II: Apparent, isolated, and dynamical horizons

Apparent horizons

- Event horizons are non-local in time, and cannot be found while evolving a spacetime
- Apparent horizons are local in time
- Apparent horizons can be used as "approximation" of event horizons
- Singularities are "always" contained in apparent horizons

Trapped surfaces

- Consider a closed 2-surface (e.g. a sphere) emitting a flash of light both inwards and outwards
- The wavefront of that flash will have an area that either increases or decreases
- The area change is called *expansion* Θ
- If $\Theta<0$ in both directions everywhere on the surface, the surface is called *trapped*; this can occur in regions of high curvature

Trapped surfaces

- Trapped surfaces indicate the presence of a singularity; see e.g. [Hawking and Ellis; Wald]
- If one of the expansions is zero, the surface is called *marginally trapped*
- A marginally trapped surface can be an *apparent horizon* if certain additional properties hold

Expansion

$$\Theta_{(\ell)} = q^{ab} D_a \ell_b$$

q_{ab} 2-metric of surface

l_a null normal of surface

Surface must be spacelike, smooth, and closed

Examples

$\Theta_{(n)} < 0, \quad \Theta_{(\ell)} > 0$ "normal" case

$\Theta_{(n)} < 0, \quad \Theta_{(\ell)} < 0$ inside black hole

$\Theta_{(n)} < 0, \quad \Theta_{(\ell)} = 0$ apparent horizon

$\Theta_{(n)} > 0, \quad \Theta_{(\ell)} > 0$ inside white hole

here l, n are the two null normals to the surface

Related concepts

- Marginal surface (MS)
- Marginally trapped surface (MTS)
- Marginally outer-trapped surface (MOTS)
- In numerical relativity, these concepts -- although well defined -- are washed together and often (wrongly) called "apparent horizon"

Apparent horizon equation in 3+1 form

- [Thornburg, Living Rev. Relativity **10**, 3 (2007)]

$$\Theta \equiv \nabla_i s^i + K_{ij} s^i s^j - K = 0,$$

s: outward pointing spacelike normal to surface
t: future pointing timelike normal to hypersurface

$$\ell^a = \frac{1}{\sqrt{2}}(s^a + t^a)$$

$$q_{ab} = \gamma_{ab} - s_a s_b$$

Solving the AH equation

The most general way to parameterize a 2-surface in a slice is to define a scalar "level-set function" F on some neighborhood of the surface, with the surface itself then being defined as the level set

$$F = 0 \qquad \text{on the surface.} \tag{3}$$

$$r = h(\theta, \phi),$$

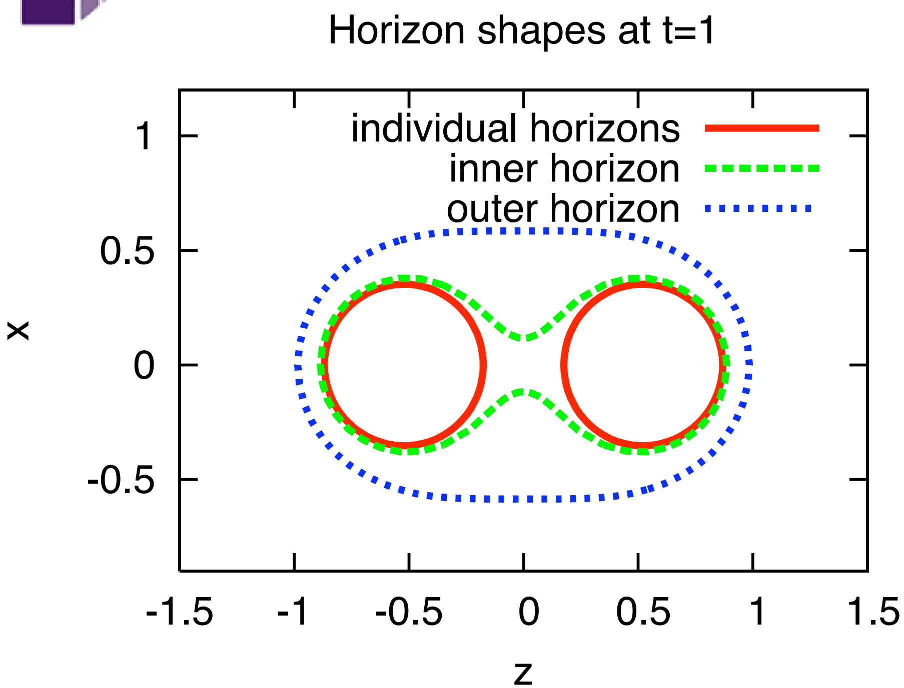
$$F \equiv r - h(\theta, \phi).$$

$$s_i = \frac{\partial_i F}{\sqrt{\gamma^{jk} \partial_j F \partial_k F}}$$

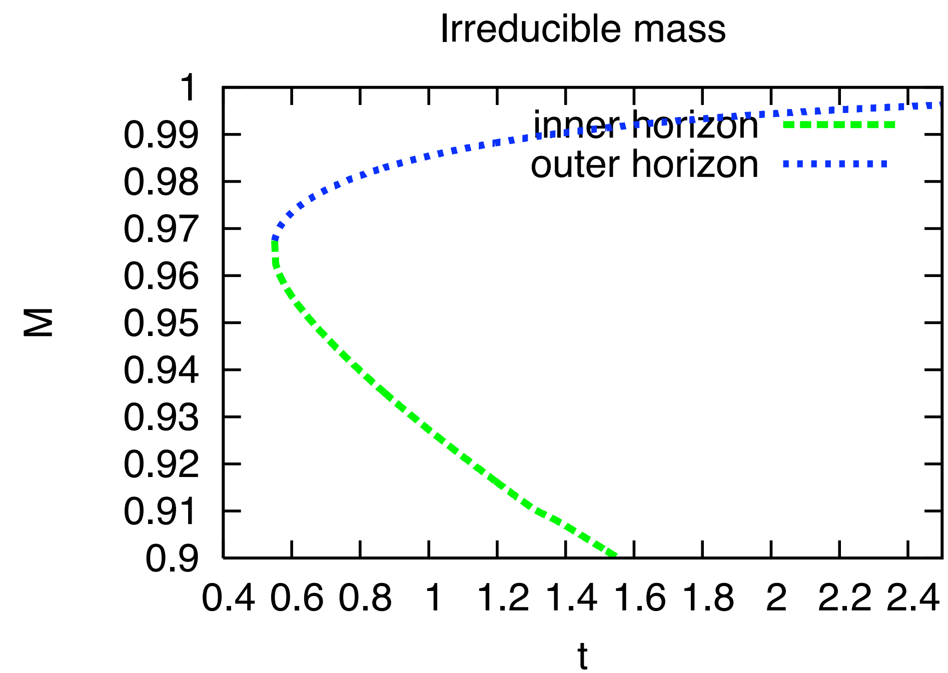
This ansatz turns the AH equation into a (nonlinear) elliptic equation for h which can be solved by standard methods

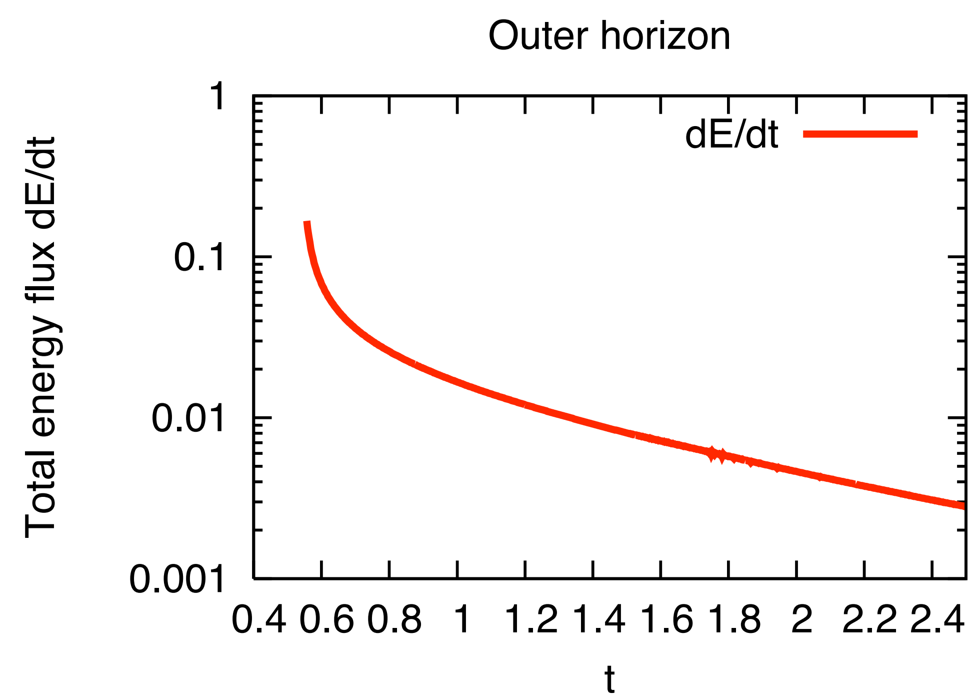
Problems

- Apparent horizons can (and do) suddenly appear, they do not merge smoothly like event horizons
- Fast elliptic solvers need a good initial guess
- Flow solvers are slow
- Coordinate singularities on the 2-surface (poles) can pose problems

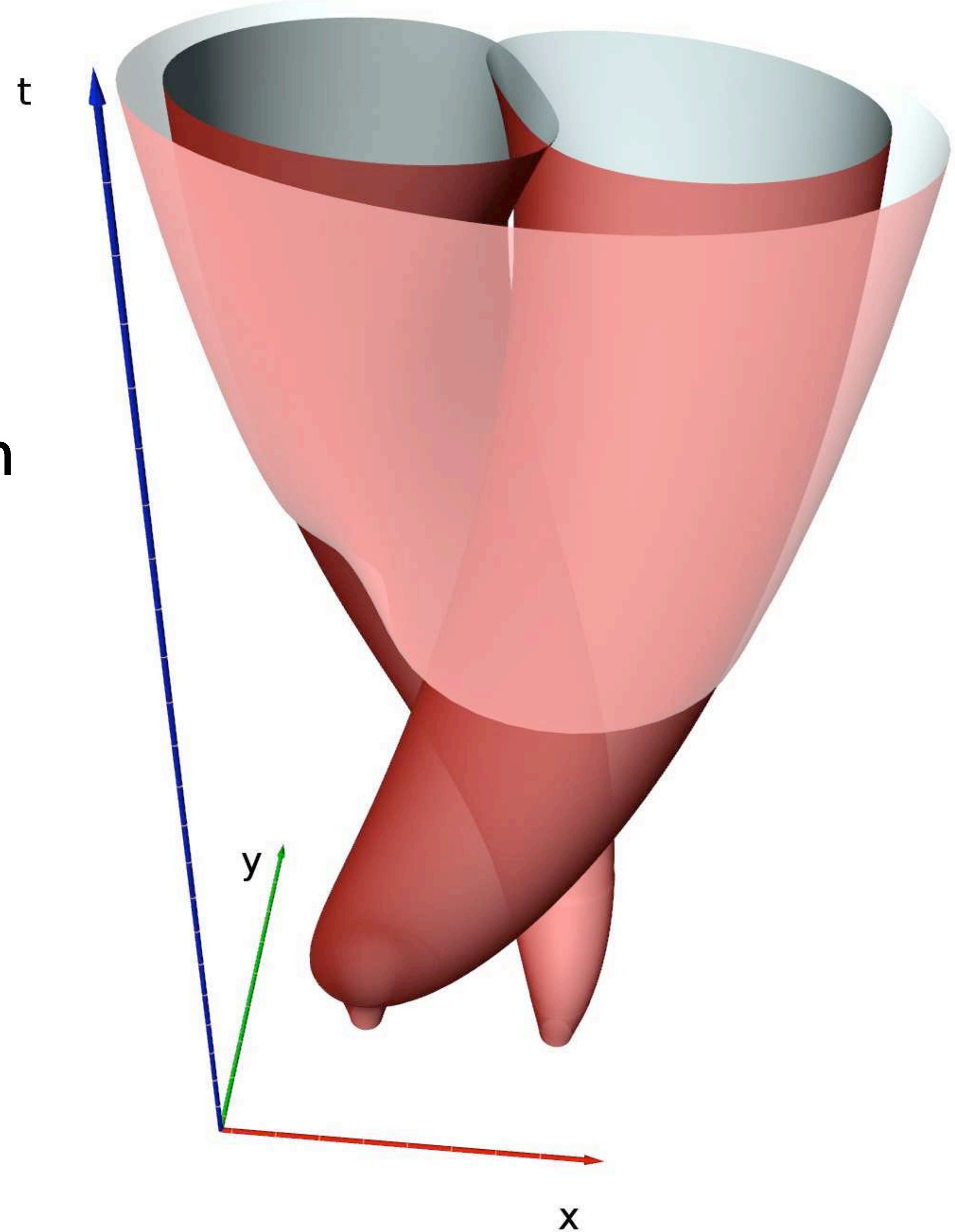


Horizon properties in a head-on collision



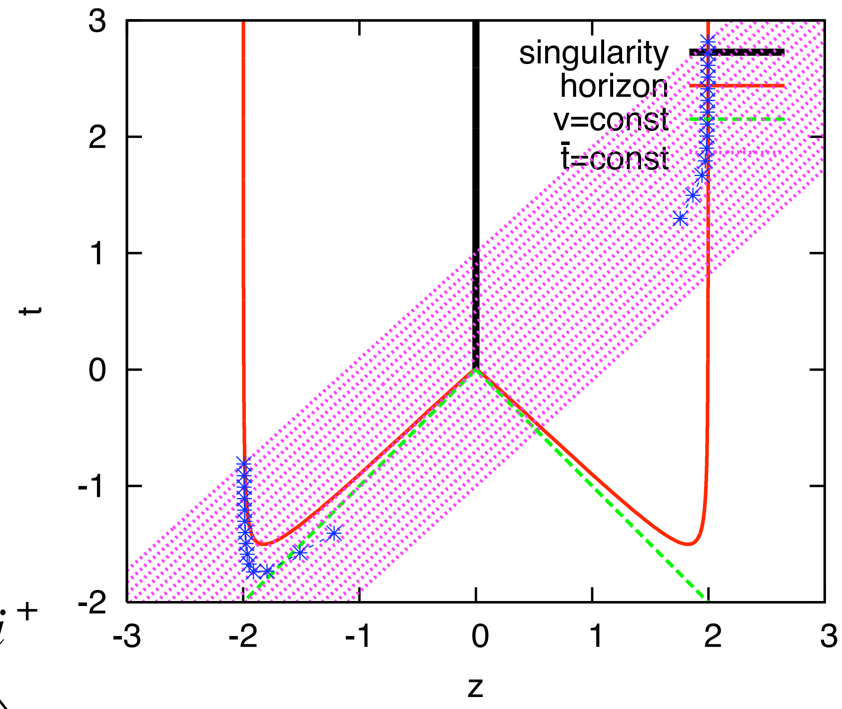


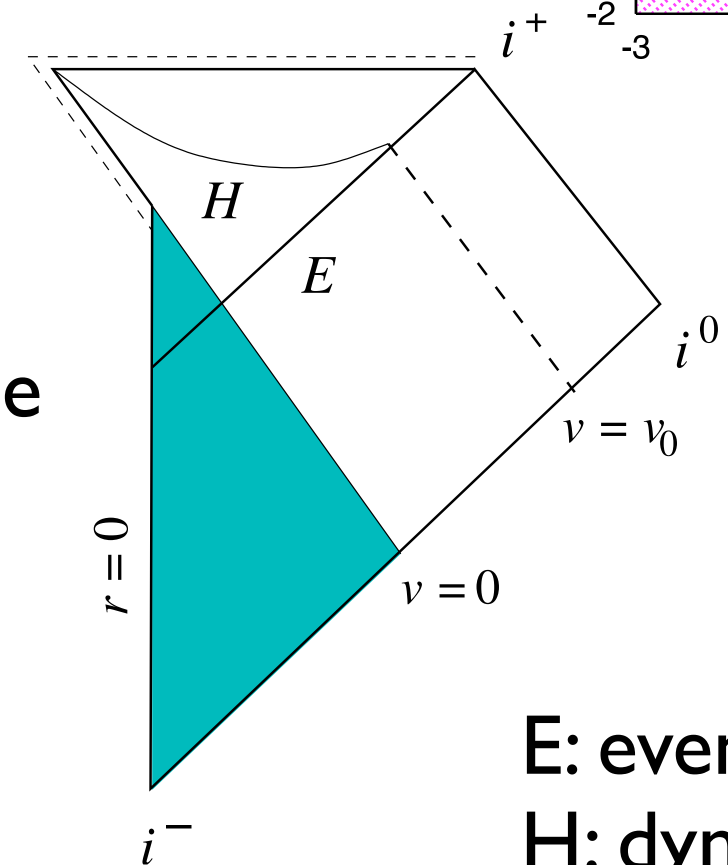
Individual and common apparent horizons in a binary black hole system



Szilágyi et al., Class. Quantum Grav. **24**, S275 (2007)

Horizons in Vaidya metric





Penrose diagram for radiation pulse of finite duration

shaded region: flat spacetime

E: event horizon

H: dynamical horizon

Apparent and event horizons

- EH is 3D null surface, AH is 2D spacelike surface
- In stationary situations AH and EH coincide
- AH depend on the foliation (i.e., the choice of time coordinate)

Mass and spin

$$2M_{\mathrm{irr}} = R = \sqrt{\frac{A}{4\pi}}$$

irreducible mass (areal radius)

$$J = -\frac{1}{8\pi} \oint K_{ij} \xi^i s^j \, d\Omega$$

angular momentum

$$M = \sqrt{M_{\mathrm{irr}}^2 + \left(\frac{J}{2M_{\mathrm{irr}}} \right)^2}$$

total mass

ξ: (generalised) Killing vector field on 2-surface

Mass and spin

- [Dreyer et al., Phys. Rev. D **67**, 024018 (2003); Ashtekar and Krishnan, Living Rev. Relativity **7**, 10 (2004); Cook and Whiting, arXiv:0706.0199 [gr-qc]]
- Angular momentum requires (generalised) Killing vector field on horizon
- Angular momentum is quasi-local, valid on every 2-surface (not just horizons), has correct ADM limit at infinity

Isolated/dynamical horizons

- If apparent horizons form a smooth world tube, then this is:
  - isolated horizon if world tube is null (stationary)
  - dynamical horizon if world tube is spacelike everywhere
  - marginally trapped tube if world tube is (partly) timelike

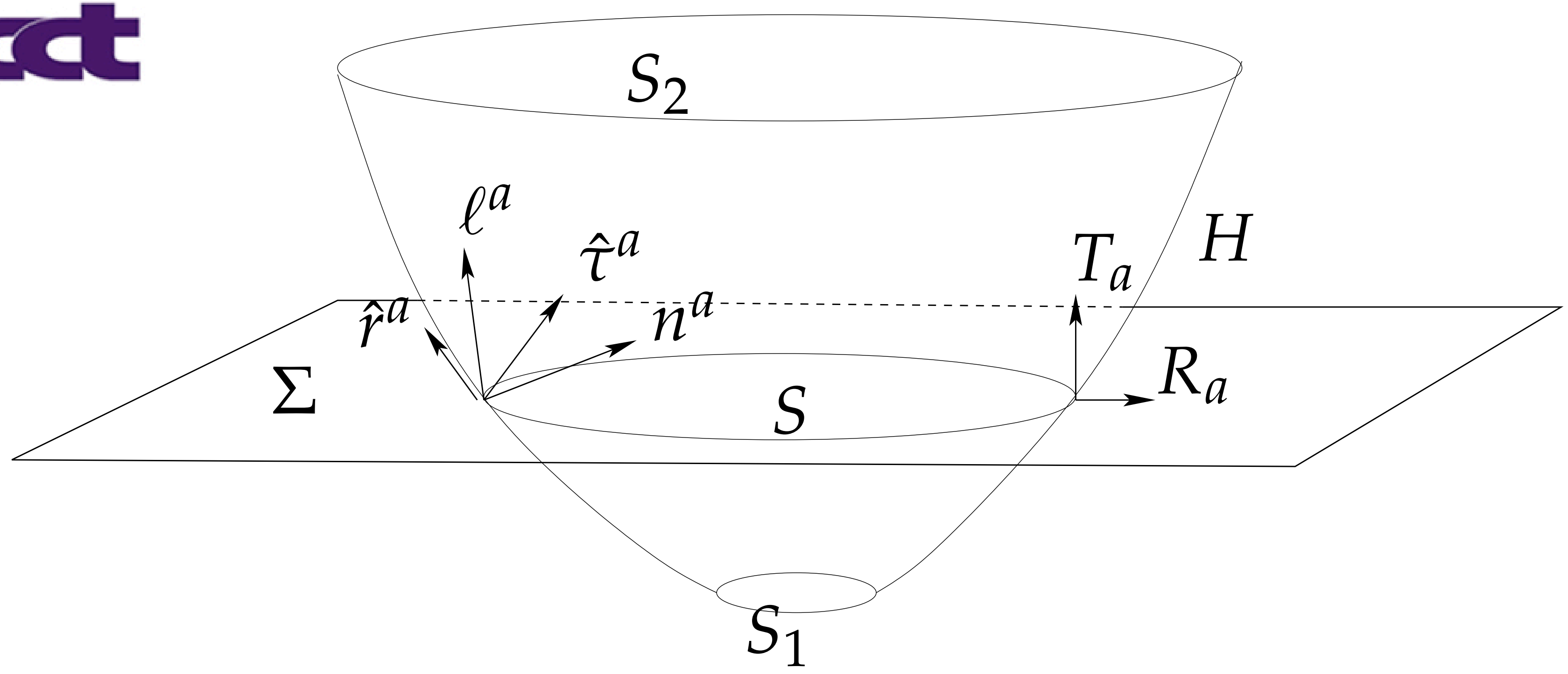


FIG. 1: A dynamical horizon H bounded by MOTSs S_1 and S_2. ℓ^a is the outgoing null normal, n^a is the ingoing null normal, $\hat{r}^a$ is the unit spacelike normal to the cross-sections, and $\hat{\tau}^a$ is the unit timelike normal to H. Σ is a Cauchy surface intersecting H in a 2-sphere S. T^a is the unit timelike normal to Σ and R^a is the unit space-like outward pointing vector normal to S and tangent to Σ.

Fluxes on dynamical horizons

- DHs have well-defined energy and angular momentum fluxes due to matter and gravitational radiation
- These fluxes have corresponding quasi-local energy and angular momentum balance laws
- Such fluxes cannot physically be defined for event horizons (since EHs grow even in flat space)

Please interrupt and ask questions at any time

Part III: Gravitational wave extraction

Why look at gravitational waves?

- Gravitational waves can be observed from far away, i.e., without sending a space probe to a black hole
- They carry information about the strong field region
- Can be detected by LIGO, GEO600, Virgo, Tama, etc.

Definition

- The concept of a gravitational wave is difficult to define in general
- Things are clear in a linearised or perturbative regime
- Things are also well-defined at future null infinity, from where we "observe" what happens in the spacetime

Methods

1. Perturbative extraction: Regge-Wheeler-Zerilli formalism
   e.g. [Pazos et al., Class. Quantum Grav. **24**, S341 (2007)]

2. Extraction based on Weyl scalars (most commonly used in numerical relativity)

Weyl tensor

- Riemann tensor contains complete curvature information (20 components in 4D)
- Ricci tensor (Einstein tensor) is determined by energy/momentum content (10 components in 4D)
- Weyl tensor (10 components) is determined by initial and boundary conditions

Weyl scalars

- Weyl tensor can be decomposed into 5 complex scalars
- Scalars are coordinate independent, but depend on a choice of (null) tetrad
- Far away, and with a good tetrad choice, these Weyl scalars have certain meanings

Null tetrad

ℓ^a	outgoing null direction
n^a	ingoing null direction
$m^a, \bar{m}^a$	tangential (complex) null directions

e.g. $\Psi_4 = C^{abcd} \bar{m}^a n^b \bar{m}^c n^d$

Interpretation

If peeling property holds: $\lim_{r \to \infty} \Psi_n \sim \frac{1}{r^{5-n}}$

Then:

Ψ_0 incoming gravitational radiation

Ψ_1 incoming gauge wave

Ψ_2 Coulomb field (mass and spin)

Ψ_3 outgoing gauge wave

Ψ_4 outgoing gravitational radiation

Implementation

- Extract waves on spheres "far away" from the source, typically 30M to 80M (only multi-domain codes do much better)
- Spheres are typically coordinate spheres
- Extract waves on several spheres at different radii and compare results
- Choose a tetrad aligned with foliation and extraction spheres

Remarks

- Extracting on several spheres allows determining whether far enough away, and allows testing peeling property (note light travel time between detectors)
- Using coordinate spheres is bad, since it makes the results gauge dependent
- Waveforms are typically decomposed into spherical harmonics

Problems

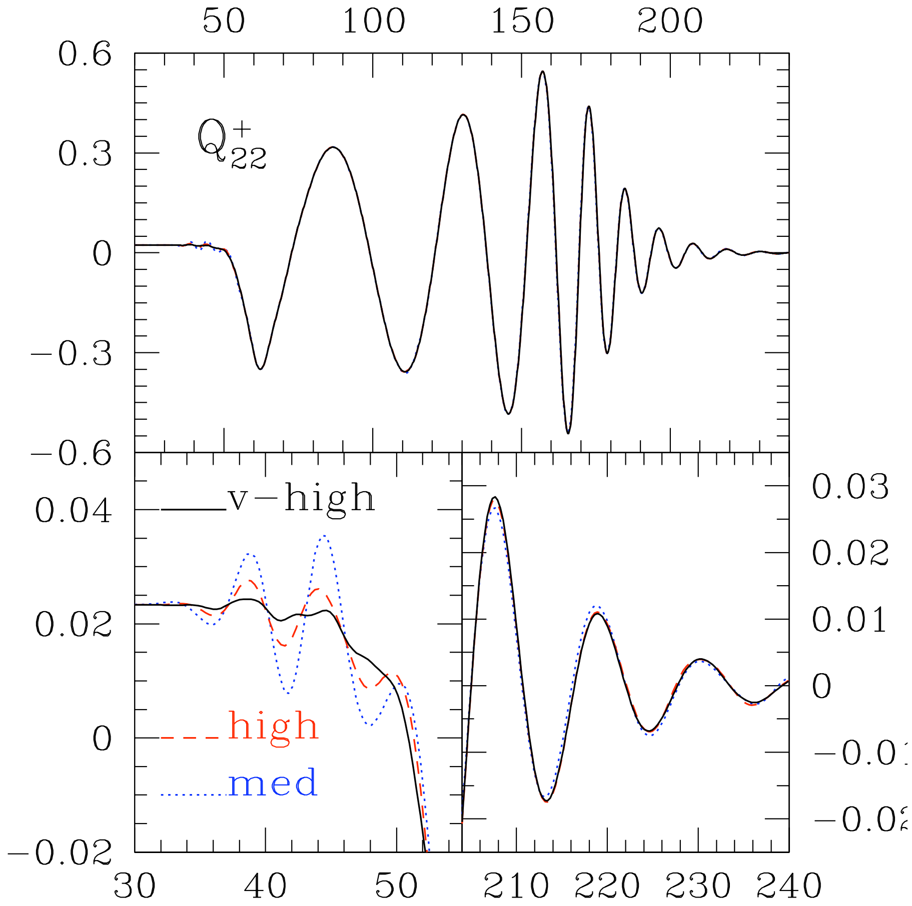
- Extracting far away is expensive:
  - Need to simulate longer (light travel time)
  - Need sufficient resolution at detector
- No obvious coordinate-independent tetrad choice on extraction spheres
- Everything is slicing dependent (may need change of time coordinate)

Proposed solutions

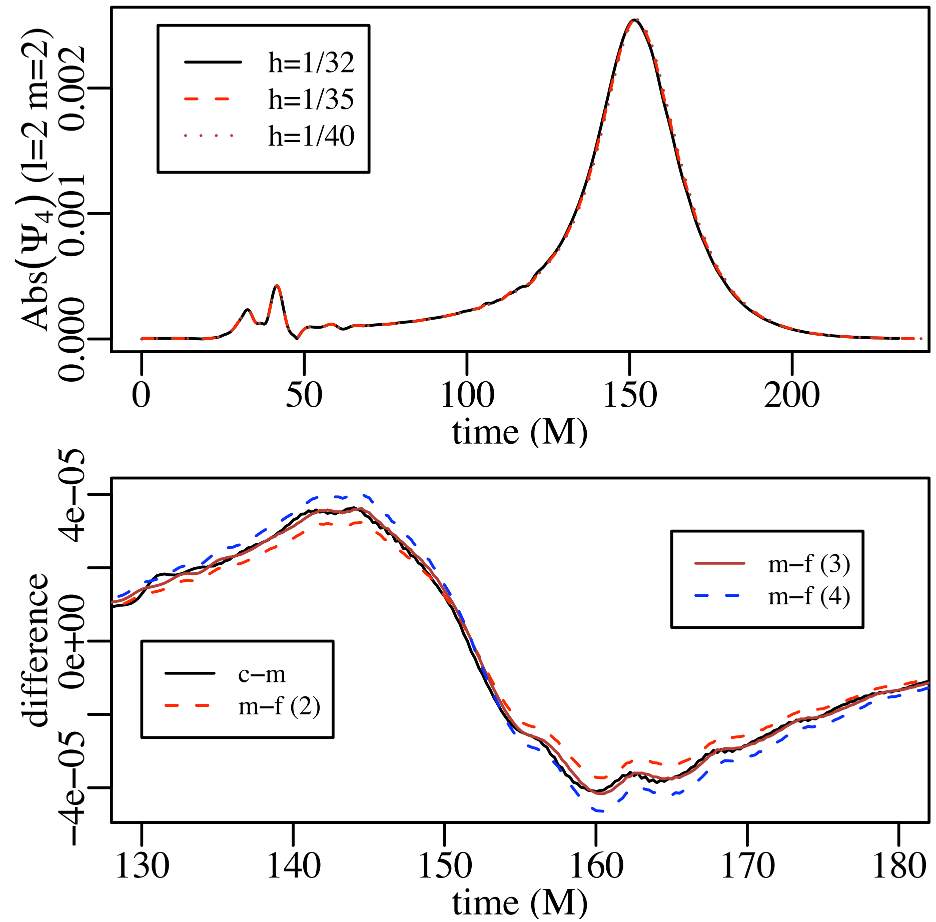
- Tetrad choice: use Kinnersley tetrad
- Surface choice: use constant expansion surfaces (gauge invariant)

Examples

[Pollney et al., arXiv: 0707.2559 [gr-qc]]



[Herrmann et al., Astrophys. J. **661**, 430 (2007)]



Alternatives

- Cauchy-characteristic matching (CCM): extract metric information at finite radius, then evolve along characteristics up to future null infinity
- Cauchy-perturbative matching (CPM): extract metric information at finite radius, then evolve perturbatively to very far away
- Could also be used for boundary conditions

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