Introduction to Numerical Relativity I

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Lectures Overview

I. The Einstein Equations
(Formulations and Gauge Conditions)

II. Analysis Methods
(Horizons and Gravitational Waves)

III. Numerical methods
(Cactus and Mesh Refinement)

CCT

- Interdisciplinary research centre at LSU, about four years old

- Computer science, phys mathematics, biology, music, ...
- Including a numerical relativity group (E. Seidel)
- http://www.cct.lsu.edu/

T. Sterling

The Einstein Equations: Formulations and Gauges

1. 3+1 Decomposition, ADM equations
2. The BSSN system and its gauge conditions
3. Handling Singularities:
   Excision, Static Punctures, Moving Punctures

Please interrupt and ask questions at any time

Part 1:
3+1 Decomposition, ADM equations

Or: How to write down the Einstein Equations such that a computer can solve them

Why solve the Einstein equations numerically?

- We want to understand gravity. The Einstein equations are complex, and analytical methods are not enough to understand them
- Astrophysical spacetimes have gravity, matter, radiation, magnetic fields, etc. Before we can study everything together, we need to understand each ingredient separately
- Gravitational wave detectors (LIGO, GEO600, ...) are taking data. Numerical calculations are necessary to understand measurements

Vacuum Equations

Einstein tensor | Energy-momentum tensor

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

"spacetime curvature" = "matter density"

$G = c = 1$ (geometric units)

$T_{\mu\nu} = 0$ (vacuum)

$\mu, \nu, \lambda, \ldots \in \{0, 1, 2, 3\}$ (spacetime indices)

Spacetime, space, and time

- The concept of "spacetime" is very elegant; it describes physics very well
- However, current numerical methods exist only for space (e.g. finite differencing) and time (e.g. Runge-Kutta integration)
- Therefore we want to decompose the spacetime into space and time, so that we can solve the Einstein equations more easily

4 = 3+1 (spacetime=space+time)

Line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

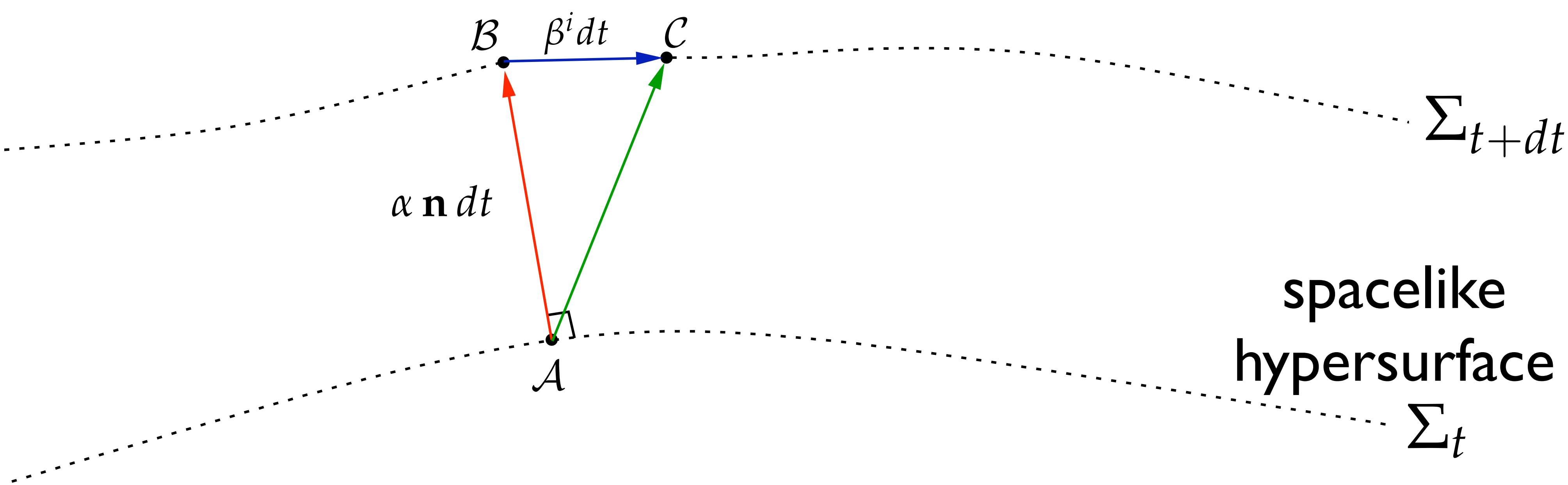
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$x^\mu = [t, x^i]$$

4-metric $$g_{\mu\nu} = \left(\begin{array}{c|c} -\alpha^2 + \beta_i \beta^i & \beta_i \\ \hline \beta_j & \gamma_{ij} \end{array}\right)$$

$$i, j, k, \ldots \in \{1, 2, 3\}$$

Lapse and Shift



α : lapse

β^i : shift

γ_{ij} : 3-metric

Foliation/Slicing of the spacetime

Foliation / Slicing

- We decompose the 4-metric into a 3-metric, lapse, and shift
- The 3-metric is positive definite; it describes a 3-dimensional (curved) space
- Lapse and shift relate the coordinate systems at different times

ADM variables

γ_{ij} 3-metric

K_{ij} extrinsic curvature $K_{ij} = -D_i n_j$

α lapse $n_\mu = \frac{D_\mu t}{|D_\mu t|}$

β^i shift

These variables, if known everywhere, describe the whole spacetime. 3-metric and extrinsic curvature describe the hypersurfaces themselves, lapse and shift describe the relation between hypersurfaces.

ADM equations

[Arnowitt, Deser, Misner (1963); York (1979)]

$$\partial_0 \gamma_{ij} = -2\alpha K_{ij}$$

$$\partial_0 K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2 K_{il} K^l_j \right)$$

$$R_{ij} = \ldots \text{second derivatives of } \gamma_{ij} \ \ldots$$

K_{ij} extrinsic curvature

$$\partial_0 = \partial_t - \mathcal{L}_\beta$$

R_{ij} 3-Ricci (curvature) tensor

Conventions

- There are different versions of the ADM quantities and ADM equations
- Note: different authors use different sign conventions, e.g. for the Ricci tensor or the extrinsic curvature
- The form presented here is "standard" in numerical relativity

Constraints

- The ADM equations correspond only to 6 (out of 10) Einstein equations.
- The other 4 Einstein equations do not contain time derivatives, i.e., they are not time evolution equations.
- Instead they are *constraints* which need to be satisfied at every time.

ADM constraints

Hamiltonian constraint

$$H := R + K^2 - K_{ij}K^{ij} = 0$$

Momentum constraint

$$M^i := D_j K^{ij} - \gamma^{ij} D_j K = 0$$

Note: no lapse and shift in constraints

Gauge conditions

- Need to specify lapse and shift in order to evolve
- Can do so (almost) freely
- Bad choices lead to bad coordinates, i.e., coordinate instabilities

Constraint evolution

- The initial condition needs to satisfy the constraints. That is, one cannot start from arbitrary data
- The Einstein evolution equation guarantee that the constraints remain satisfied if they are satisfied initially
- This allows *unconstrained evolution*

Monitoring evolutions

- While a computer calculates a spacetime, it is necessary to check whether the result is good
- One important method is to examine how well the constraints are satisfied
- Another important test is to check for high-frequency noise (zig-zags) in the solution (which should not be there)

Comparison to Electrodynamics

- The ADM equations are somewhat similar to the Maxwell equations
- The Maxwell equations contain constraints which relate charge density to E and B fields
- The Maxwell equations also contain time evolution equations which describe waves

Review

- We have decomposed the Einstein spacetime equations into spatial and temporal equations
- We have decomposed the 4-quantities into 3-quantities and scalars
- In principle, we can now solve these equations as one solves the Maxwell equations

Caveat

- However, this is not yet so
- The ADM equations are unstable; time evolutions do not work: Noise is amplified, constraints grow
- Proving or examining this is complex since the equations are non-linear
- Years of experiments led to two stable formulations: BSSN and Harmonic

Please interrupt and ask questions at any time

Part II: BSSN equations, gauge conditions

Or: How to write down the Einstein Equations such that things actually work

BSSN

- The BSSN equations are an extension of the ADM equations
- The BSSN equations have a long history; many people contributed to this formulation
- BSSN: Baumgarte-Shapiro-Shibata-Nakamura
- Earlier: also Nakamura, Oohara, Kojima
- Later: also Alcubierre and others

Some References

- BSSN equations: [Alcubierre et al., Phys. Rev. D **62**, 044034 (2000)]
- BSSN gauges: [Alcubierre et al., Phys. Rev. D **67**, 084023 (2003); Baker et al., Phys. Rev. Lett. **96**, 111102 (2005)]
- Short overview: [Pollney et al., arXiv:0707.2559 [gr-qc]]

BSSN variables

conformal factor	$\phi = \ln\Psi = \frac{1}{12}\ln\gamma\,,$
trace K	$K = K^i_i = \gamma^{ij}K_{ij}\,,$
conformal metric	$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}\,,$
traceless extrinsic curvature	$\tilde{A}_{ij} = e^{-4\phi}A_{ij}\,. \quad A_{ij} \equiv K_{ij} - \frac{1}{3}\gamma_{ij}K\,,$
Gamma	$\tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} = -\tilde{\gamma}^{ij}_{\;\;,j}\,,$

(BSSN evolution equations not shown)

Meaning of the BSSN variables

- ϕ: closely related to Hamiltonian constraint
- K: gauge part in extrinsic curvature
  (K is determined by choice of lapse α)
- Γ: gauge part in 3-metric
  (Γ is determined by choice of shift β)

Why does BSSN use these variables?

- Historically, the BSSN system was found through trial and error. It is much better than the ADM system
- These days we know that the BSSN system is well-posed (in a certain sense)
- The BSSN system also damps constraint violations created by numerical errors

Additional BSSN constraints

In addition to the ADM constraints,
the BSSN system places certain conditions
onto its variables:

$$\begin{aligned} \tilde{\gamma} &= 1 \\ \tilde{A}_i^i &= 0 \\ -\tilde{\gamma}^{ij}_{,j} &= \tilde{\Gamma}^i \end{aligned}$$

These constraints need to be monitored, and
the first two can also be enforced

Gauge conditions

- A gauge condition chooses the coordinate system for the numerical result
- Bad gauge conditions can lead to instabilities and must be avoided
- Often, gauge conditions are specified as choices for lapse and shift

Simple gauge conditions

$\alpha = 1$ geodesic slicing (unstable)

$\Box t = 0 \quad [\partial_t \alpha = \ldots]$ harmonic slicing

$K = 0 \quad [\Delta \alpha = \alpha R]$ maximal slicing (expensive)

$\beta^i = 0$ normal coordinates (problematic near horizons)

$"\Gamma^i = 0" \quad [\Delta \beta^i = \ldots]$ minimal distortion (expensive)

1+log slicing

- Similar properties to maximal slicing
- Similar to harmonic slicing
- Idea: instead of enforcing K=0, only drive K towards zero
- Advantages: hyperbolic instead of elliptic (much faster)
- Currently best known lapse choice for BSSN

1+log slicing

K evolution equation for zero shift:

$$\partial_t K = -\Delta\alpha + \alpha(R + K^2)$$

desired behaviour for K:

$$"\partial_t K = -C\,K + \cdots"$$

1+log slicing:

$$\partial_t \alpha = -\alpha^2 f(\alpha)(K - K_0)$$

Other (improved) variants of 1+log slicing exist.

Gamma-driver shift

- Similar properties to minimal distortion (idea: choose shift such that metric distortion is minimised)
- Drive distortion to zero, same as 1+log drives K to zero
- Advantages: hyperbolic instead of elliptic (much faster)
- Currently best known shift choice for BSSN

Gamma-driver shift

$$\partial_t \beta^i - \beta^j \partial_j \beta^i = \frac{3}{4} \alpha B^i ,$$
$$\partial_t B^i - \beta^j \partial_j B^i = \partial_t \tilde{\Gamma}^i - \beta^j \partial_j \tilde{\Gamma}^i - \eta B^i ,$$

- B: new variable: time derivative of shift
- η: coefficient for driving time scale
- drives distortion to zero

Important 1+log and Γ-driver properties

- Symmetry seeking: Coordinates become stationary in stationary spacetimes
- Hyperbolic, not elliptic, i.e., require only additional time evolution equations
- Driver conditions which act on a certain time scale (determined by a parameter)

Review

- ADM equations don't work, but BSSN equations do
- Need to specify gauge condition (prescription for lapse and shift) before evolution
- There are well-known good gauge conditions for the BSSN system

Please interrupt and ask questions at any time

Part III:
Singularities: excision and moving punctures

Or: How to handle black holes

Black Holes

- Black holes are interesting since they are stationary solutions of the Einstein equations
- Unfortunately, black holes have singularities
- We need numerical methods to handle these singularities
- Fortunately, these singularities are hidden (?) behind and event horizon

Possible singularity handling methods

1. Use a foliation which avoids the singularity
2. Place an internal boundary around the singularity (excision)
3. Factor out the singularity analytically (static punctures)
4. Accept constraint violations inside the horizon (moving punctures)

Singularity avoiding foliations

- Probably earliest attempt
- Choose initial data which do not contain the singularity (e.g. a collapsing star before a singularity has formed)
- Choose a slicing condition which avoids the singularity (e.g. maximal slicing, 1+log)
- Does not work well in practice (end state is not stationary)

Excision

- One of the two working methods
- Idea: Information from inside the black hole does not influence the exterior, hence we don't need to know the interior -- cut it away
- The boundary is outgoing, i.e., there is no boundary condition needed

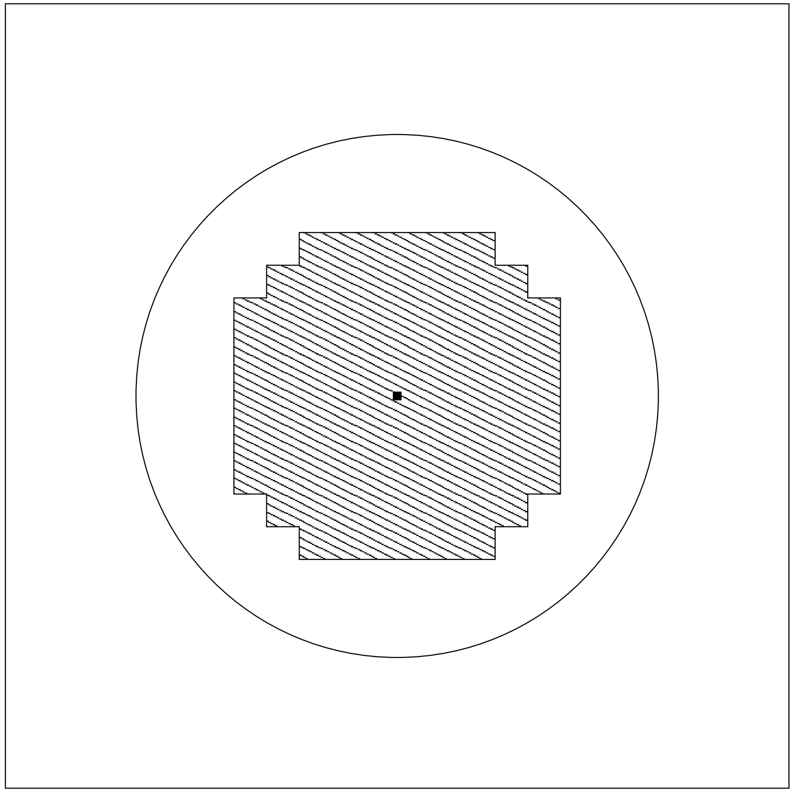
Excision

- Need to locate horizon to know where to excise: use apparent horizon instead of event horizon (see Lecture II)
- See e.g. [Sperhake et al., Phys. Rev. D **71**, 124042 (2005); Pretorius, Phys. Rev. Lett. **95**, 121101 (2005)]

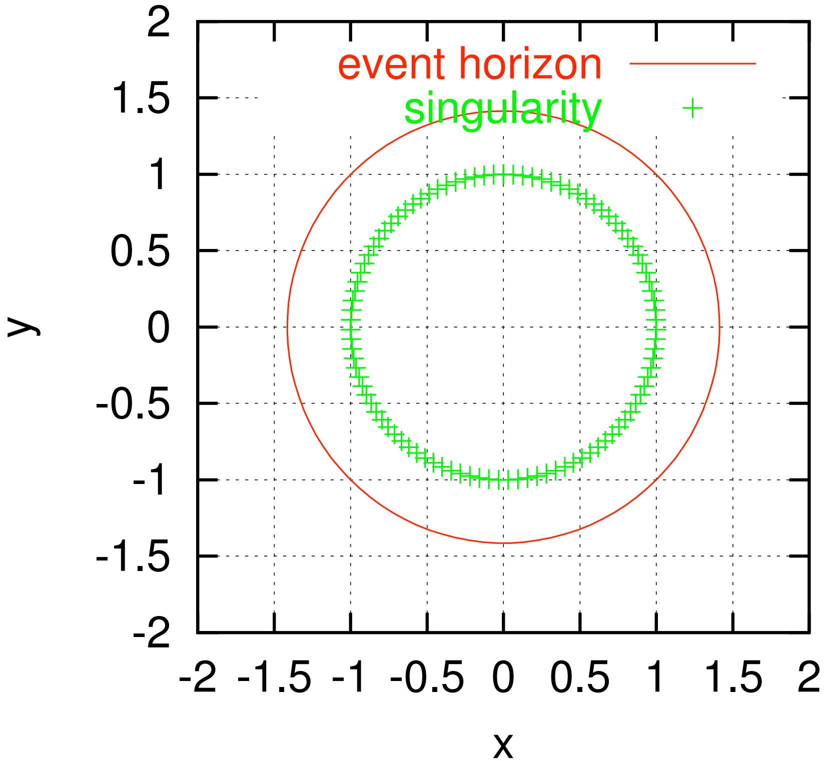
Kerr-Schild with excision

rotating black hole, M=1, a=1

grid cell layout



horizon and singularity



Problems with excision

- In Cartesian coordinates, the boundary has an irregular shape
- Near the boundary one needs to use one-sided derivatives or extrapolation
- “Simple excision” excises a regular-shaped box, but cannot handle large black hole spins
- If the black hole moves, the excision boundary has to move with it

Static Punctures

- Precursor to moving punctures
- Idea: Factor out the singularity analytically and evolve only the regular remainder numerically
- Requires a gauge that keeps the singularity at a fixed location
- See e.g. [Alcubierre et al., Phys. Rev. D **67**, 084023 (2003)]

Static Punctures

- Choose initial data with singular metric
- Decompose metric: $\gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij}$
- Keep Ψ constant in time, evolve only rescaled metric
- Set α=0 at puncture to ensure that ψ remains static there
- Stagger singularity between grid points

Why static punctures work

- ψ and its derivatives are calculated analytically and are therefore accurate
- This even increases the overall accuracy of a simulation
- Everything else remains well-behaved near the singularity and therefore accurate

Disadvantages

- Requires special initial data
- Need to re-write all evolution equations
- Gauge conditions need to be updated
- Requires comoving coordinate systems

Moving Punctures

- The other working method
- An extension of static punctures
- Idea: Instead of handling the singular conformal factor ψ analytically, handle it numerically
- See e.g. [Campanelli et al., Phys. Rev. Lett. **96**, 111101 (2006); Baker et al., Phys. Rev. Lett. **96**, 111102 (2006); Brown et al., arXiv:0707.3101 [gr-qc]]

Problems with moving punctures

- Not obvious why this should work (but it does)
- Conformal factor becomes inaccurate near singularity due to numerical errors (but this is inside the horizon)
- Constraints are violated near the puncture (but this does not affect the exterior)

Advantages of moving punctures

- Uses standard BSSN equations and gauges
- No restrictions on initial data or coordinate systems
- “Just works”

Review

- Singularities are "everywhere" and need to be handled correctly
- The (currently known) two working methods are excision and moving punctures
- Excision introduces inner boundary conditions
- Moving punctures accept constraint violations in the horizon

Please interrupt and ask questions at any time

Part IV: Movies

Or: How it looks when put together

Movies

- [show binary black hole collision movie]
- [show gravitational wave movie]

Sources for this presentation

- Some equations and figures taken from:
  - C. D. Ott, PhD thesis, Universität Potsdam
  - E. Schnetter, PhD thesis, Universität Tübingen
  - D. Pollney et al., arXiv:0707.2559 [gr-qc]