Feedback control of acoustic musical instruments: Collocated control using physical analogs

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Traditionally, the average professional musician has owned numerous acoustic musical instruments, many of them having distinctive acoustic qualities. However, a modern musician could prefer to have a single musical instrument whose acoustics are programmable by feedback control, where acoustic variables are estimated from sensor measurements in real time and then fed back in order to influence the controlled variables. In this paper, theory is presented that describes stable feedback control of an acoustic musical instrument. The presentation should be accessible to members of the musical acoustics community who may have limited or no experience with feedback control. First, the only control strategy guaranteed to be stable subject to any musical instrument mobility is described: the sensors and actuators must be collocated, and the controller must emulate a physical analog system. Next, the most fundamental feedback controllers and the corresponding physical analog systems are presented. The effects that these controllers have on acoustic musical instruments are described. Finally, practical design challenges are discussed. A proof explains why changing the resonance frequency of a musical resonance requires much more control power than changing the decay time of the resonance.
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I. INTRODUCTION

A. Background

Musicians have been playing musical instruments for many thousands of years. In the traditional performance paradigm, a musician employs feedback to help him or her control the mechanical excitation that he or she provides to a musical instrument. Consider the interaction model in Fig. 1, which is adapted from Bill Verplank's interaction design sketchbook.[1] We note that other researchers in the field of computer music employ diagrams that emphasize the separation of the musical instrument interface from the sound synthesizer;[2–4] however, we prefer our more general depiction here because it emphasizes the physical, musical interaction.

We assume for convenience that the musician uses his or her hand to provide a mechanical excitation to the musical instrument. We acknowledge that a wide range of other excitations could of course be provided using the mouth, the remainder of the body, etc., but we do not explicitly depict them. Since the musician can change the mechanical impedance of his or her hand, we include a filter in the mechanical excitation path that is modulated by the state of the hand as shown in Fig. 1.

The musician relies heavily on auditory feedback. However, visual and haptic feedback (see Fig. 1) also play an important role. For example, a musician can employ them to help orient his or her hands on a piano keyboard before pressing any keys. Figure 1 further includes the brain as the link in the composite feedback system where the visual, auditory, and haptic modalities are combined. The brain and the hand are internally interconnected by the human motor system control loop.[5]

B. Motivation

Traditionally, the typical musician has owned many different kinds of musical instruments. The advantage has been that he or she could achieve a certain characteristic sound by playing the corresponding musical instrument. However, transporting the multitude of musical instruments to performances has been onerous, and musicians have required large amounts of space to store all of the musical instruments.

Rather than requiring different musical instruments in order to achieve different characteristic sounds, some musicians would prefer to have a single musical instrument with *programmable acoustics*. To achieve different or new sound characteristics, the musician could simply load a different musical instrument program. Similarly, to construct a novel musical instrument, the musician could write a new "acoustical" program rather than going through the time-consuming process of mechanically constructing a new musical instrument from scratch. Furthermore, besides obviating the need to store large numbers of musical instruments, the interface to the musical instrument would remain similar

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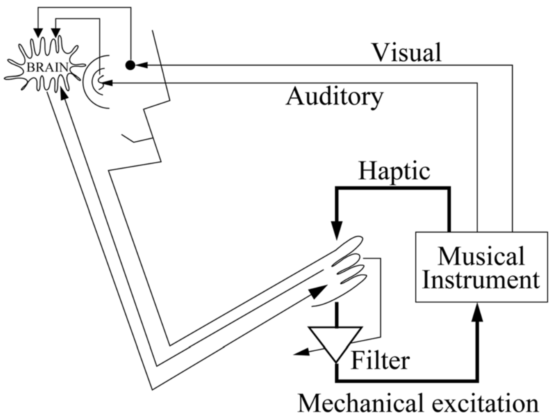

FIG. 1. Musician interacting with a musical instrument.

despite changes in the acoustical behavior. This *interface constancy* would alleviate the musician of the time-consuming process of learning to play via different human-instrument interfaces merely to obtain different sound characteristics.

While the introduction of Musical Instrument Digital Interface (MIDI), keyboards hasprovided for a common interface to sound synthesis algorithms since the 1980s, feedback controlled acoustic musical instruments offer an additional advantage over MIDI and related keyboards by retaining an important characteristic of traditional acoustic musical instruments: *direct haptic control of sound*. Due to the mechanical feedback nature of the external, thickly drawn, loop in Fig. 1, haptic control provides a musician with intimate control of tangible sound at the audio sampling rate. For instance, as with traditional chordophones, idiophones, and membranophones, the musician can stop a feedback controlled acoustic musical instrument from vibrating by placing his or her hand in direct contact with a vibrating portion of the musical instrument. In other words, the musician can change the mechanical impedance load that the he or she places on the musical instrument, which is represented by the hand-adjustable filter in Fig. 1. Furthermore, a physical acoustic object vibrates according to an infinite number of modes, and with direct haptic control, a musician can physically and intimately interact with all of these modes in real time at any points in the acoustic medium. The modes and their sound are tangible.

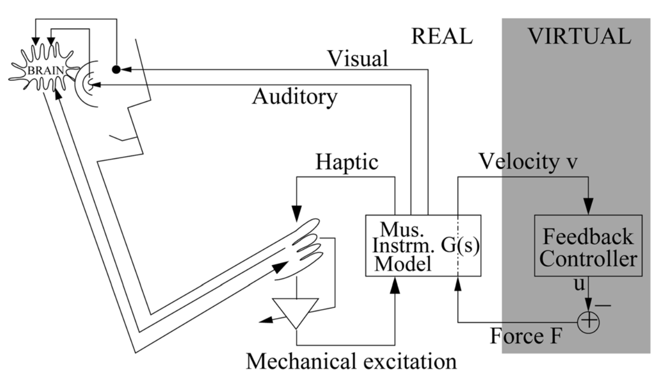

FIG. 2. Musician interacting with a feedback controlled acoustic musical instrument.

C. Feedback control of an acoustic musical instrument

We suggest employing digital feedback control of the acoustical characteristics of a musical instrument in order to make them programmable by computer. In other words, we suggest adding a new feedback loop to form the interaction model shown in Fig. 2, which links the real environment together with a virtual, programmable environment. The virtual environment can fundamentally change the natural dynamics of the real musical instrument. The musical instrument serves as an *augmented reality interface*, where the real acoustic medium is augmented by elements in the virtual environment.[6] In the terms the categorization of Miranda *et al*., the musical instrument is an augmented musical instrument.[2]

II. FEEDBACK CONTROL APPROACH

A. Overview

A feedback control system is capable of driving that which it controls, such as a musical instrument, unstable. In this situation, the controller continues to add energy to the musical instrument causing it to self-oscillate. These oscillations can grow and become unpleasantly loud like the howling sound of an unstable public address system. For this reason, we argue that a feedback controller for an acoustic musical instrument should be *capable of stable control* even if it is not always operated in this mode. The capability of stable control ensures that the mechanical design supports a wide variety of musical instrument dynamics. In addition, the controller can always be reprogrammed later to make oscillations in the control system grow, should this be desired, for instance for sustaining string vibrations.[7,8]

To facilitate analysis of the control system stability, we redraw the diagram from the perspective of the controller as shown in Fig. 3. One or more sensors (not shown) provide the feedback controller with an observation of the musical instrument state. For example, let v represent the velocity of some part of the musical instrument at a single point. For instance, this could be the velocity of a point along a vibrating string, the velocity of a point on a membrane, etc. The output signal u from the controller is negated to form the signal F, which is applied to the musical instrument by an actuator. For example, let F be a force exerted on the musical instrument at the same point where the velocity is measured.[9] Then $G(s)$, as shown in Figs. 2 and 3, is the Laplace-domain transfer function of the mechanical mobility of the musical instrument model as seen from the sensor and

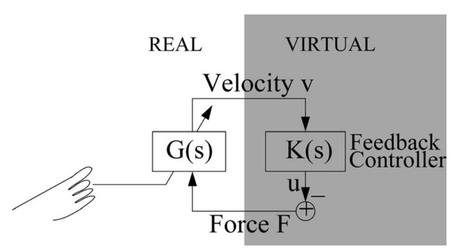

FIG. 3. Feedback loop shown from controller's perspective, with the controller represented by the Laplace-domain transfer function $K(s)$.

actuator. In this paper, we focus on linear control only, so the controller is represented by $K(s)$ (see Fig. 3).

Over time as the musician plays notes, he or she alters parameters of the musical instrument, such as the fundamental frequency. As a consequence, $G(s)$ can change drastically in a short period of time, for instance while the musician changes notes. This important consideration is illustrated in Fig. 3 by the arrow from the hand to the block $G(s)$.

Let us assume for a moment that we could use a method such as fundamental frequency detection to estimate $G(s)$ in real time as it changes.[10] The musical instrument could theoretically be controlled using the pole placement technique, where the poles of the musical instrument are moved to certain desired locations (see pp. 533–543 of Ref. [11]). However, $G(s)$ would be known only approximately, so it would be difficult to apply pole placement successfully to move the poles of the musical instrument by any appreciable distance. Furthermore, in a practical configuration, estimating $G(s)$ fast and accurately enough would be difficult. For instance, fundamental frequency estimators usually wait for at least one period of vibration before providing an estimate,[10] but for a note at 100 Hz, this delay is already 10 ms. During this time, the feedback controller could well have destabilized the musical instrument due to an outdated estimate of $G(s)$.

B. Control strategy

As a consequence of the above argument, we believe that the control system should be designed so that *the control system is stable no matter what $G(s)$ is.*

Let us assume for convenience that at least a little energy leaks out of the musical instrument over time at all frequencies. In other words, the musical instrument model $G(s)$ is "dissipative." Then the only way to ensure stability is to implement collocated control and to choose $K(s)$ so that the effect of the feedback control system is equivalent to coupling the musical instrument to a *passive physical analog* system (see Appendix A). In this case, the control system cannot continuously add any energy to the vibrations of the musical instrument, implying that the net control system must be stable. We now explain how to design such a control system.

1. Physical analogs

The feedback controller should model the motion of a virtual, passive mechanical system coupled to the musical instrument, where F represents the force that would be exerted on the musical instrument by the virtual mechanical system if the point at which they were coupled moved at velocity v. Specific examples of passive virtual mechanical systems are provided in Sec. III. Other analogs are also possible including electrical circuit analogs.[12]

On the other hand, if $K(s)$ does *not* represent the dynamics of a passive physical analog system, then there exists a $G(s)$ corresponding to a dissipative musical instrument model that will drive the control system unstable. A proof of this lesser known fact is available in the literature on force-feedback control of robots.[13] Consequently, *to ensure stability, $K(s)$ must represent a physical analog system, such as a mechanical system, an electrical system, etc.*

2. Collocated control

We now introduce the concept of collocated control. Consider what happens if one displaces a spring from its rest position: it responds immediately with a force. Now consider if the musical instrument is displaced at its sensor position. If the feedback control system is to simulate a spring, then it must respond immediately at that position with a force. However, this can only happen if an actuator is placed at the same position as the sensor. Otherwise if the actuator is placed some distance away from the sensor, the actuated force must travel through the acoustic medium before it reaches the sensor, introducing a time delay.

Hence, in order for the feedback control system to be capable of behaving like a physical analog system, it must be capable of responding immediately, which implies that each sensor should be complemented with an actuator at the same position in space. In other words, *each sensor and actuator pair should be collocated.*

For more information on how to collocate a sensor and an actuator, see Sec. IV A.

3. Implications

We have argued that the control system must be stable if the sensors and actuators are collocated, $K(s)$ corresponds to a physical analog system, and $G(s)$ is dissipative. Appendix A provides a mathematical framework for determining if a candidate rational function $K(s)$ corresponds to a physical analog system. This framework implies that $K_0K(s)$ corresponds to a physical analog system as long as $K(s)$ also corresponds to a physical analog system, and the control gain $K_0 \geq 0$. Consequently, *in theory, the system remains stable even for arbitrarily large control gains K_0.*

This remarkable property is known as unconditional stability,[14,15] and it implies that *the control system will be endowed with a large amount of control authority over the acoustic musical instrument*. This property is why we emphasize that musical instrument designers creating feedback controlled acoustic musical instruments should ensure that sensors and actuators are collocated.

C. Sensor and actuator placement

We have argued that each sensor and actuator pair should be collocated, but there is another important consideration when choosing where to place such a pair. If a pair is placed at a node for a certain resonance, then the pair will not be able to the control that resonance. The resonance is formally considered both *uncontrollable* because the actuator cannot actuate it and *unobservable* because the sensor cannot sense it (see pp. 559–572 of Ref. [11]). As a consequence, *care should be taken that the resonances to be controlled vibrate with appreciable amplitude at the position where the collocated sensor and actuator are placed.*

III. FUNDAMENTAL CONTROLLERS

A. Model of a musical resonance

In order to study the effects of feedback control on an acoustic musical instrument, we need to first introduce a

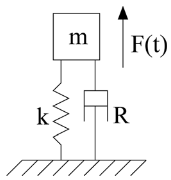

FIG. 4. Model for musical instrument resonance with equivalent mass m, damping parameter R, spring constant k, and external force $F(t)$.

model to serve as $G(s)$. We employ the most parsimonious model possible that can resonate like a musical instrument, which is a single resonance. At first inspection, it might seem more prudent to employ a model incorporating many resonances; however, we have showed that this is unnecessary when studying feedback control of what we call "musical resonances." Musical resonances have decay times that are much longer than the period of oscillation. The behavior of each musical resonance subject to feedback control is independent of the others. Hence, it is sufficient to study the effects of feedback control on a single musical resonance (see pp. 43–50 of Ref. [16]).

The mechanical physical analog model for a musical resonance consists of an equivalent mass m in kg, an equivalent damper with damping constant R in N/(m/s), and an equivalent spring with stiffness k in N/m (Ref. 17). $R > 0$ ensures that the musical resonance dissipates energy over time at all frequencies, but to ensure that the resonance is musical, the damping R must be small enough so that the period of oscillation is much shorter than the decay time constant. Mathematically expressed, $R \ll \sqrt{km}/\pi$ (Ref. 18).

As shown in Fig. 4, these elements all move at the same speed and form what is also known as the damped harmonic oscillator. If an external force F is applied to the oscillator, then a differential equation describing the dynamics of the forced oscillator may be written as

$$m\ddot{x} + R\dot{x} + kx = F = -u, \tag{1}$$

where x is the position in m, $\dot{x}$ is the velocity in m/s, and $\ddot{x}$ is the acceleration in m/s^2. Assuming that R is small enough for the oscillator to be lightly damped, it resonates at the fundamental frequency

$$f_0 \approx \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{2}$$

Because the resonance is a musical resonance, its decay time

$$\tau = 2m/R \tag{3}$$

is much longer than its period

B. Proportional-integral-derivative (PID) control

Proportional-integral-derivative (PID) control can be easily employed because of its simplicity. The feedback force is a sum of the scaled versions of a measured variable (P), its integral (I), and its derivative (D) (see pp. 236–248 of Ref. [11]). If we measure the velocity and the actuator is collocated with the sensor, then the controller does not change the model order. However, to ensure that the differential equation for the motion of the musical resonance retains its same form, we write the feedback Eq. (4) in terms of the displacement x instead. There is a term proportional to the displacement (P_P), a term proportional to the derivative of the displacement (P_D), and a term proportional to the second derivative of the displacement (P_{DD}):

$$u \overset{\Delta}{=} P_{DD}\ddot{x} + P_D\dot{x} + P_P x. \tag{4}$$

The feedback control law can also be expressed in the Laplace domain, where we assume zero initial conditions for convenience. Introducing the Laplace transform pairs $\dot{x}(t) \leftrightarrow V(s)$, $u(t) \leftrightarrow U(s)$, and $F(t) \leftrightarrow F(s)$, we can write the transfer function for the feedback controller

$$K_{PID}(s) = \frac{U(s)}{V(s)} = P_{DD}s + P_D + P_P/s. \tag{5}$$

Substituting Eq. (4) into the dynamics for a musical resonance Eq. (1), we obtain a differential equation describing the closed-loop system:[19]

$$(m + P_{DD})\ddot{x} + (R + P_D)\dot{x} + (k + P_P)x = 0, \tag{6}$$

which illustrates that the feedback control parameters allow independent adjustment of the equivalent mass, equivalent damping parameter, and quivalent spring constant. When these values are substituted into Eqs. (2) and (3), we obtain the resonance frequency with control $\hat{f}_0$ and the decay time constant with control $\hat{\tau}$:

$$\hat{f}_0 \approx \frac{1}{2\pi}\sqrt{\frac{k + P_P}{m + P_{DD}}} \tag{7}$$

and

$$\hat{\tau} = \frac{2(m + P_{DD})}{R + P_D}. \tag{8}$$

Consider PID control of a vibrating string, as depicted in Fig. 5 (top), where $K(s)$ is the controller. The microphone symbol represents the string velocity sensor, and the loudspeaker symbol represents the string actuator. Figure 5 (bottom) illustrates the mechanical analog of the controller. According to Eq. (6), the controller is equivalent to attaching a point mass $P_{DD} \geq 0$, damper $P_D \geq 0$, and spring $P_P \geq 0$ to the string at the point where the sensor and actuator are collocated. The existence of the mechanical analog containing only passive elements is proof that PID control itself is passive. It can also be shown that $K_{PID}(s)$ is passive using the technique outlined in Appendix A.

P_P affects only the resonance frequency, and P_D affects only the decay time, while P_{DD} affects both of the resonance frequency and decay time. Hence, P_P and P_D can serve as convenient and independent control parameters.

In practice, most acoustic musical instruments have multiple resonances. The effect of the control is qualitatively

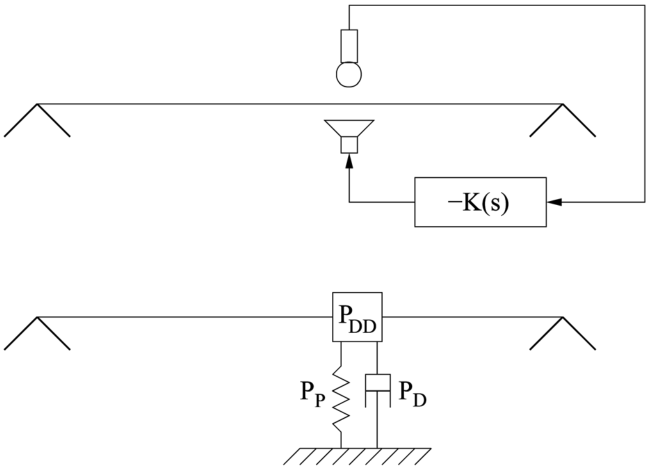

FIG. 5. Control of a vibrating string with a collocated sensor (depicted as microphone) and actuator (depicted as loudspeaker) (top); Mechanical analog of PID control (bottom).

similar, except that there are not enough control parameters to individually adjust each of the musical instrument resonances. Increasing P_D causes the decay time of all of the observable modes to decrease. In contrast, the P_P displacement feedback term has a low-pass characteristic with respect to the P_D velocity feedback term. Therefore, P_P affects the low frequency modes more than the high frequency modes. In practice, the P_P term brings about a noticeable resonance frequency shift of primarily the lowest, significantly dominant resonance. Thus, P_P changes the pitch of the musical instrument, but it can also detune the harmonic series.

C. Controller phase response

The phase response $\angle K(s)|_{s=j2\pi f} = \angle K(j2\pi f)$ indicates what the control effect will be at frequency f in Hz, and the magnitude response $|K(j2\pi f)|$ indictates how strong this effect will be (see pp. 51–52 of Ref. [16]).[17] Consider an acoustic musical instrument with a musical resonance at any frequency f_i.

(1) *If at frequency f_i the control signal u leads the velocity $\dot{x}$ by $-\pi/2$ rad, then the resonance frequency f_i increases under control. If the magnitude of the feedback signal at f_i is increased, then f_i will increase further*. We know this fact from PID control with $P_{DD} = 0$ and $P_D = 0$. Then $K_{PID}(s) = P_P/s$ with $P_P > 0$, so $\angle K_{PID}(j2\pi f) = -\pi/2$ rad for $f > 0$.

(2) *Similarly, if at frequency f_i the control signal u leads the velocity $\dot{x}$ by $\pi/2$ rad, then the resonance frequency f_i decreases under control*.

(3) *If the control signal u is precisely in phase with the velocity $\dot{x}$, then the decay time of the resonance is decreased*. This case corresponds to the PID damping with $P_D > 0$ while $P_P = 0$ and $P_{DD} = 0$.

(4) *If the control signal u is precisely π rad out of phase with the velocity $\dot{x}$, then the controller will increase the decay time of the resonance of drive it unstable*. For large enough gains, the musical instrument will be driven unstable because the controller is not passive.[7,8]

(5) *If at frequency f_i, $\angle K(j2\pi f_i)$ does not match any of the preceding cases, both the resonant frequency and damping of f_i will be changed.*

The above criteria in conjunction with the physical analog approach can be employed to design many more controllers such as bandpass filters, notch filters, etc., which are described next.

D. Bandpass control

A bandpass filter controller has the advantage that primarily only the frequency region of interest will be affected by control. We employ the following bandpass filter as a controller:

$$K_{bp}(s) = K_0 \frac{\frac{2\pi f_c s}{Q}}{s^2 + \frac{2\pi f_c s}{Q} + (2\pi f_c)^2}. \quad (9)$$

The frequency response of the bandpass filter with center frequency $f_c = 200$ Hz, quality factor $Q = 20$, and control gain $K_0 = 1$ is shown in Fig. 6.[20] The effect of the controller can be determined chiefly by looking at the phase response (see Fig. 6, bottom). Any musical instrument resonances located very near f_c will be damped. Musical instrument resonances located somewhat near to f_c will be damped, but also shifted some in frequency. Resonances far away from f_c will be chiefly unaffected by the controller. The mechanical physical analog of Eq. (9) is shown in Fig. 7 in the context of controlling a vibrating string. The existence of the physical analog proves that $K_{bp}(s)$ is passive for $Q > 0$ and $f_c > 0$.

Multiple bandpass filters can be placed in parallel in the signal processing chain for the purpose of controlling multiple musical resonances in this fashion. Each filter can be employed to obtain nearly independent control of a musical resonance. For instance, a bandpass filter can be tuned to the first, third, fifth, etc., harmonics of a vibrating string. Then

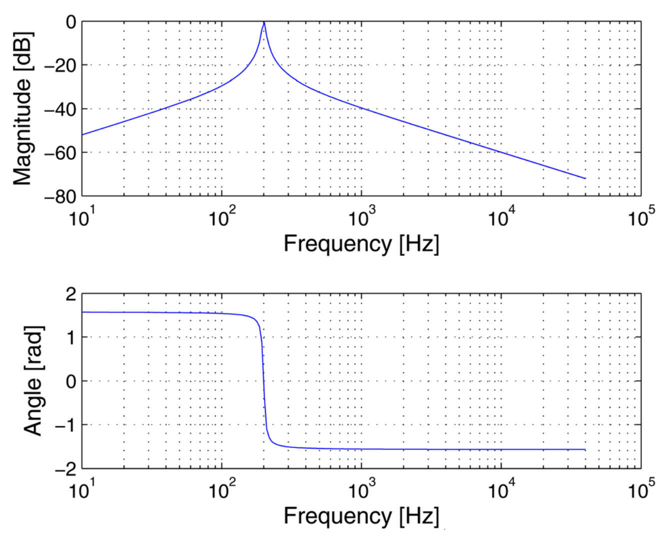

FIG. 6. (Color online) Frequency response of the bandpass controller given by Eq. (9) with $Q = 20$ and $f_c = 200$ Hz.

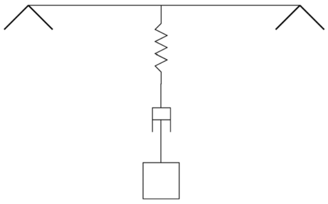

FIG. 7. Mechanical analog of the bandpass controller.

the controller can make the musical instrument sound as if it is being played an octave higher because it damps the odd harmonic musical resonances. The multiple bandpass filter configuration corresponds to multiple physical analogs of the type in Fig. 7 all being attached to the musical instrument at the same point along the string (not shown).

E. Notch filter control

A notch filter controller performs the inverse function of the bandpass controller-it applies damping over all frequencies, except for those in the region near the center frequency f_c. We suggest employing the following notch filter:

$$K_{notch}(s) = K_0 \frac{s^2 + \frac{2\pi f_c s}{\alpha \tilde{Q}} + (2\pi f_c)^2}{s^2 + \frac{2\pi f_c s}{\tilde{Q}} + (2\pi f_c)^2}. \tag{10}$$

The frequency response is shown in Fig. 8 for $f_c = 300$ Hz, $\tilde{Q} = 1$, $K_0 = 1$ and $\alpha = 10$. $\tilde{Q}$ is not precisely the quality factor,[21] but it has the same qualitative interpretation: larger $\tilde{Q}$'s correspond to "tighter" magnitude response notches. Figure 9 shows the mechanical physical analog of the notch controller in the context of controlling a vibrating string.

The notch filter controller can be employed to the musical resonances of the musical instrument decay quickly, with the exception of any musical resonances lying significantly far inside the notch. The controller causes somewhat unnatural closed-loop sounding behavior as most musicians are

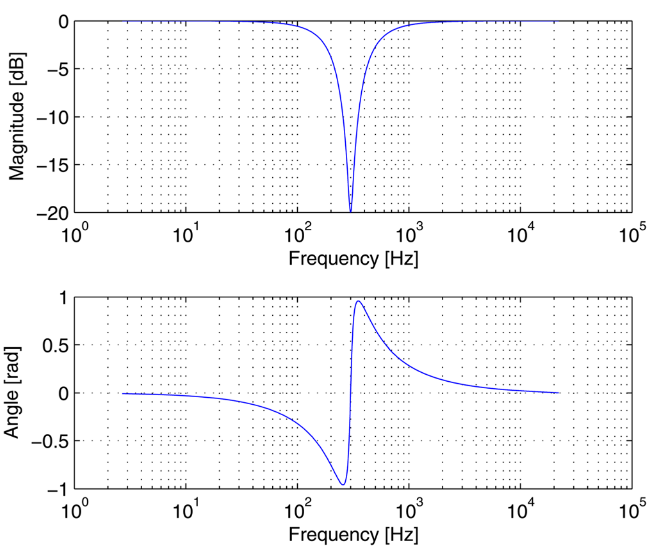

FIG. 8. (Color online) Frequency response of the notch filter controller given by Eq. (10) with $f_c = 300$ Hz, $\tilde{Q} = 1$, and $\alpha = 10$.

unaccustomed to musical instruments having a small number of dominant resonances. Note that multiple notches can be created at different frequencies by building up $K(s)$ out of multiple instances of Eq. (10) placed in series.

F. Other formally passive controllers

Many other passive controllers are possible. They can be obtained by connecting together arbitrary networks of masses, springs, and dampers (or equivalently capacitors, inductors, and resistors), as well as gyrators and transformers, and solving for $K(s)$. They can also be obtained using techniques introduced by Otto Brune.[12] Berdahl's thesis incorporates a study of the effects of a higher order alternating-filter type controller for simultaneously shifting the resonance frequencies of multiple resonances (see pp. 55–62 of Ref. [16]). By employing networks of collocated sensors and actuators, it is possible to generalize the passivity concept to multiple-input multiple-output (MIMO) control systems (see pp. 64–71 of Ref. [16]). Their qualitative effect is similar, but for the sake of brevity, we do not discuss them further here.

G. Physical analogs with inverted gain

Although this work focuses primarily on feedback control using passive physical analogs, it is worthwhile to consider what happens if the physical analogs are non-passive. In this case, a dissipative musical instrument exists that will be driven unstable given sufficiently large loop gain.[13] However, for a specific dissipative musical instrument model, the control system will still be stable for sufficiently small loop gains. For instance, if the feedback controller behaves like a negative stiffness, the control system can still be stable if the musical instrument is stiff enough to counteract the negative stiffness.

We do not belabor the point, but we include these effects below in Table I, which summarizes the effects of all of the aforementioned fundamental controllers. A check mark in the IG, or inverted gain, column indicates that the sign of the control gain is inverted. The sign inversion shifts the phase of the control signal by π radians (180°), rendering the controller non-passive but nevertheless useful in some contexts.

Now that we have provided some insight into the effects of the fundamental feedback controllers on a musical instrument, we discuss some challenges in the control system design.

IV. CHALLENGES

A. Matched sensor and actuator pairs

Based on our experience, we feel that it is necessary to specify the sensors and actuators at the earliest stages of the design. Then the remainder of the musical instrument can be constructed sensibly around the sensors and actuators. Nevertheless, we cannot stress enough that researchers or experimenters who have limited experience in transducer design should study prior designs before embarking on new projects.

As discussed in Sec. II B 2, in order to implement feedback control with physical analogs, each sensor and actuator pair needs to be collocated. However, in practice, neither a real sensor nor a real actuator operates precisely at a point.

TABLE I. Summary of the primary effects of the fundamental controllers on the musical instrument.

Controller	IG	Primary effect on musical instrument
PID control $P_P > 0, P_D = 0, P_{DD} = 0$		Increase the resonance frequency of primarily the lowest musical resonances
PID control $P_P < 0, P_D = 0, P_{DD} = 0$	✓	Decrease the resonance frequency of primarily the lowest musical resonances
PID control $P_P = 0, P_D > 0, P_{DD} = 0$		Decrease the decay time of musical resonances
PID control $P_P = 0, P_D < 0, P_{DD} = 0$	✓	Increase the decay time of musical resonances
PID control $P_P = 0, P_D = 0, P_{DD} > 0$		Decrease the resonance frequency and increase the decay time of primarily the higher musical resonances
PID control $P_P = 0, P_D = 0, P_{DD} < 0$	✓	Increase the resonance frequency and decrease the decay time of primarily the higher musical resonances
Bandpass control $K_{bp}(s)$		Decrease the decay time of musical resonances near the center frequency f_c
Bandpass control $-K_{bp}(s)$	✓	Increase the decay time of musical resonances near the center frequency f_c
Notch filter control $K_{notch}(s)$		Decrease the decay time of musical resonances away from the center frequency f_c
Notch filter control $-K_{notch}(s)$	✓	Increase the decay time of musical resonances away from the center frequency f_c
Pseudo-velocity control $K_I(s)$		Decrease the decay time of primarily the lowest musical resonances
Pseudo-velocity control $-K_I(s)$	✓	Increase the decay time of primarily the lowest musical resonances

Instead, they operate over finitely sized spatial windows of the medium being controlled. *If the sensor and actuator geometry are the same, in other words, if they both operate over the same spatial window, then the sensor and the actuator are considered "matched" and are effectively collocated.*[22]

An example of a perfectly matched sensor and actuator would be a pair of identical ideal piezoelectric bending patches on either side of an infinitely thin membrane. With the membrane oriented horizontally, the patch on top of the membrane would be placed directly over the patch on the underside of the membrane.[22] *From the controller's perspective, the elements of the physical analog simply act over the spatial window rather than at a single point.*

B. Control system delay

Any delay in the feedback loop makes it impossible for the feedback control to perfectly emulate a physical analog system (see Sec. II B 2). For this reason, it is important to minimize the delay due to the implementation of $K(s)$. In fact, even though it may seem old-fashioned, in some cases it can be easier to use analog filters, or digital signals can even be employed to change the coefficients of analog filters in real time.[17] However, in many cases where the musical instrument will be involved in live music performance, it can be more convenient if $K(s)$ is implemented approximately using a digital controller.

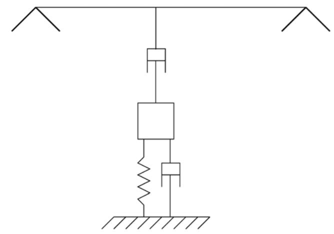

FIG. 9. Mechanical physical analog of the notch controller.

Figure 10 (left) represents the ideal transfer function $K(s)$ that the digital controller approximates. The digital controller is represented in Fig. 10 (right). The fastest kind of digital controller operates as follows:

(1) An analog-to-digital converter (ADC) samples the input signal and immediately sends the input sample to a digital controller (T_A seconds).
(2) The digital controller computes a single output sample according to the new input sample to achieve the effect of $K(s)$. However, the controller requires T_C seconds to compute the output sample.
(3) A digital-to-analog converter (DAC) converts the output sample from a digital representation into an analog voltage, requiring an additional T_D seconds.
(4) Finally a zero-order-hold (ZOH) circuit brings the output voltage to the new output value, and the process repeats from step 1.

The nature of the ZOH delays the control signal by $T_Z = (1/2f_S)$ seconds, where f_S is the sampling rate in samples/s (see pp. 666–668 of Ref. [11]). It follows that the digital controller actually implements the transfer function

$$e^{-sT_A} e^{-sT_C} K(s) e^{-sT_D} e^{-sT_Z} = e^{-sT_T} K(s), \quad (11)$$

where the total delay $T_T = T_A + T_C + T_D + T_Z$. *This delay contributes to significant phase lag at higher frequencies, which in practice prevents the controller from perfectly emulating a physical analog system at high frequencies.*

As of the current date, appropriate hardware for implementing sample-by-sample feedback control at common audio sampling rates is not widely available; however, some usable hardware can be obtained. For instance, one can employ a standard microcontroller or motor controller if it has fast ADC inputs. This hardware is portable, but the output is typically produced by a pulse width modulator (PWM)

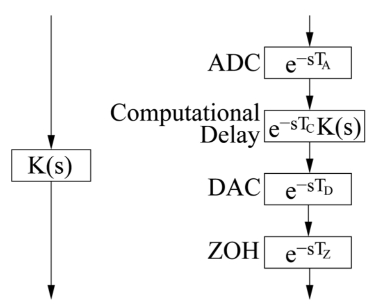

FIG. 10. Ideal representation of controller (left); digital approximate implementation of controller (right).

rather than a ZOH, and consequently, the sound may suffer from aliasing. In contrast, a good hardware solution for laboratory experiments is currently to obtain a general purpose desktop computer running Linux, install a data acquisition card, and setup the Real-Time Application Interface.[23] Using this method, we were able to achieve $T_T \approx 24\ \mu s$ for $f_S = 40$ kHz by reserving one of the processor cores for emulating $K(s)$ and doing nothing else.[24]

One might wonder if it is possible to compensate for the delay by carefully adjusting $K(s)$. Although some degree of compensation is possible over a small frequency band using a lead, it is not possible to compensate over the entire frequency band—the maximum phase compensation of a lead is bounded (see pp. 436–447 of Ref. [11]). Otherwise, if $K(s)$ could perfectly compensate for the delay, then $K(s)$ would be capable of predicting the future! Similarly, it is not possible to perfectly compensate for control system bandwidth limitations as described in the following section.

C. Control system bandwidth

The bandwidth of the control system is always limited by lowpass filters. For example, the amplifier that powers the actuator cannot respond infinitely quickly, so it must contain at least one lowpass filter, and the sensing circuit is usually bandlimited to avoid passing along too much sensing noise to the actuator. Sometimes it is also necessary to insert an additional filter after the ZOH to decrease the sound of aliasing. Ideally, one should know what all of the bandwidth-limiting lowpass filters are, but in practice, it is usually sufficient to know what the lowest lowpass filter cutoff frequency is, as long as it is at least an octave or two beneath those of any of the other lowpass filters.

Let us consider a bandwidth limit of f_B that is due to a lowpass filter whose cutoff frequency is much lower than those due to any other lowpass filters in the feedback controller. Let us assume also that A_B is the maximum gain at low frequencies for which the practical, implemented system is stable given bandwidth limit f_B. Then it holds approximately that the feedback control system has a gain-bandwidth product of $A_B f_B$ (see pp. 96–97 of Ref. [16]). In other words, if one desires to have a larger loop gain $\check{A}_B$ in order to obtain a stronger control effect on the dynamics of the musical instrument, then the bandwidth limit must be decreased to $\check{f}_B$, where

$$\check{A}_B \check{f}_B \lesssim A_B f_B. \tag{12}$$

$\check{f}_B$ can be decreased by reducing the cutoff frequency of the lowest lowpass filter. Hence, there is clearly a trade-off between control system bandwidth and maximum stable loop gain. This implies that *in practice, it is easier to control vibrations at lower frequencies than at higher frequencies.*

D. Control power

Consider a resonance with a long decay time relative to its period. We call this a "musical resonance." It turns out that *it is easier in practice to employ feedback control to change the decay time of the musical resonance than to change its resonance frequency.*

To prove this fact, we consider what would happen if we were to provide the musical resonance in Fig. 4 with an impulsive excitation, while restricting the control energy applied over all time to be constant. If we compare increasing the frequency of vibration by η_P versus decreasing the decay time by the factor η_D, we discover that

$$\eta_P \approx \frac{\eta_D}{2Q}, \tag{13}$$

where Q is the quality factor of the resonance (see Appendix B for the derivation)

In other words, if the amount of control power is fixed, the maximum decrease in the decay time η_D is about $2Q$ larger than the maximum increase in resonance frequency η_P! To help put this result in perspective, we provide another equation relating the decay time τ, the fundamental frequency f_0, and the Q, which holds for $Q \gg 1$ (Ref. 25):

$$Q \approx \pi \tau f_0. \tag{14}$$

We consider the concrete example from a low G (G2) note on an acoustic guitar whose lowest resonance frequency is about 100 Hz. A realistic decay time for the uncontrolled string is $\tau = 2.5$ s. The resonance is musical, so its $Q \gg 1$, and in fact by Eq. (14), $Q \approx 800$. To decrease the decay time by a factor of 8, we would choose $(1+\eta_D) = 8$. Applying Eq. (13), we have

$$\eta_P \approx \frac{\eta_D}{2Q} = \frac{7}{2(800)} \approx 0.0044. \tag{15}$$

Hence, using the same amount of control energy to change the resonance frequency instead of changing the decay time, we could increase the resonance frequency by only

$$\frac{0.0044}{2^{1/12}-1} \approx \frac{0.0044}{0.0595} \approx 7 \text{ cents}, \tag{16}$$

which is 7/100 of the logarithmic distance in frequency between two adjacent notes on a piano keyboard. In an approximate sense, this is on the order of the smallest perceptible change in fundamental frequency for short durations.[26] This example underscores why we argue that it is more difficult to change the frequency of a musical resonance than to change its decay time using feedback control.

When designing a new musical instrument, it might be of interest for the controller to be capable of changing the fundamental frequency of a musical instrument. The result from this section implies that doing so by controlling vibrations in the acoustical medium is difficult. Instead, it is better for the controller to directly actuate that which changes the wavespeed in the medium, such as actuating the tension of a membrane, etc.

V. CONCLUSIONS

Using collocated feedback control, it is easier to change the decay time of a musical resonance than to change its resonance frequency. Other practical design challenges include achieving sufficiently large control bandwidth, minimizing the control delay, and constructing matched sensor and actuator pairs. However, as long as musical instrument designers understand these challenges, they can take them into account while creating novel feedback controlled musical instruments. These musical instruments provide for direct haptic control of the sound, and the acoustics of the musical instruments can be significantly altered by feedback control, providing programmable acoustics while preserving the form of the human-instrument interface.

Nonlinear feedback control can serve as an extension to linear feedback control. Some limitations of nonlinear control are similar, implying the utility of analysis and design using physical analogs.[16] However, a more comprehensive nonlinear analysis beyond physical analogs would be challenging yet enlightening, so we hope to make it the subject of future work which would describe more exotic dynamic behaviors than can be provided by linear systems alone.

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APPENDIX A: POSITIVE REAL CONTROLLER DESIGN FRAMEWORK

We provide a mathematical framework for determining whether a physical analog system exists that corresponds to a controller $K(s)$. It is not necessary to derive the physical analog system itself. Instead, it turns out that it is sufficient to check the phase response of $K(s)$. Also, rather than restricting $G(s)$ to represent a musical resonance as in the prior sections, we now allow $G(s)$ to represent the "matched window" driving point admittance or impedance of a musical instrument over any particular certain window.

A. Positive real functions

Positive real functions were introduced in 1931 for synthesizing transfer functions corresponding to electrical analog circuits.[12] Since then, a rational function $\tilde{K}(s)$ has usually been defined to be *positive real* if and only if $\tilde{K}(s)$ is real when s is real, and $Re\{\tilde{K}(s)\} \geq 0$ for all s such that $Re\{s\} \geq 0$. Similarly, a rational function $\tilde{G}(s)$ has usually been defined to be *strictly positive real* if $\tilde{G}(s+\epsilon)$ is positive real for all real $\epsilon > 0$ (Ref. 27).

However, for our purposes it is much more convenient to use the following equivalent definitions in terms of the angle along the frequency axis. We define the rational function $\tilde{K}(s)$ to be *positive real* if and only if $|\angle\tilde{K}(j2\pi f)| \leq (\pi/2)$ for all frequencies f, and similarly the rational function $\tilde{G}(s)$ is *strictly positive real* if and only if $|\angle\tilde{G}(j2\pi f)| < (\pi/2)$ for all frequencies f (Ref. 27).

For instance, the bandpass and notch controllers in this paper are strictly positive real since the magnitude of the phase responses shown in Figs. 6 and 8 (bottom) does not exceed $(\pi/2)$. Now we explain some of the further properties of positive real and strictly positive real functions:[28]

(1) $1/\tilde{K}(s)$ is positive real.
(2) $1/\tilde{G}(s)$ is strictly positive real.
(3) If $\tilde{K}(s)$ represents either the driving point impedance or driving point mobility of a system, then the system is *passive* as seen from the driving point.
(4) If $\tilde{G}(s)$ represents either the driving point impedance or driving point mobility of a system, then the system is *dissipative* as seen from the driving point.
(5) $\tilde{K}(s)$ and $\tilde{G}(s)$ are stable.
(6) $\tilde{K}(s)$ and $\tilde{G}(s)$ are minimum phase.
(7) The relative degrees of $\tilde{K}(s)$ and $\tilde{G}(s)$ must be less than 2.
(8) If a point sensor and a point actuator operate on a musical instrument and are collocated, then both $V(s)/F(s)$ and $F(s)/V(s)$ are strictly positive real because there must be some damping at all frequencies in a dissipative musical instrument.
(9) If a *matched* sensor and actuator operate on a musical instrument and are collocated, then both $V(s)/F(s)$ and $F(s)/V(s)$ are strictly positive real because there must be some damping at all frequencies in a dissipative musical instrument.
(10) No matter what causal time-domain function $f(t)$ is used to excite the driving point, the velocity response $v(t)$ will be such that $\int_0^\infty f(t)v(t)dt \geq 0$.

We have purposefully chosen $\tilde{G}(s)$ and $\tilde{K}(s)$ from this appendix to correspond with the controller transfer function $K(s)$ and the musical instrument transfer function $G(s)$. Hence from Sec. II B 3, we have that the control system will be stable if the controller $K(s)$ is passive (i.e., positive real) and $G(s)$ is dissipative (i.e., strictly positive real).

APPENDIX B: ANALYSIS OF CONTROL POWER

A. Approach

In this section, we show that changing the resonance frequency of a musical resonance requires significantly more control energy than changing its decay time. The practical implications of this statement are explained in Sec. IV D. Let us consider controlling the resonance modeled by the mass, spring, and damper from Sec. III A. For a second-order system such as this, the quality factor

$$Q \triangleq \frac{k}{R(2\pi f_0)} = \frac{m(2\pi f_0)}{R}, \tag{B1}$$

where f_0 is the uncontrolled frequency of vibration. From Eq. (14), the decay time for a musical resonance is approximately $Q/\pi \gg 1$ periods. We will show that if the amount of control power is fixed, the maximum decrease in the decay time is about $2Q$ times larger than the maximum increase in resonance frequency as expressed in Eq. (13)!

We present an example using displacement (P_P) and velocity (P_D) feedback to independently control the resonance frequency and the decay time. In order to find the impulse response of the controlled system, we assume an impulse $\delta(t)$ is added to the force signal. In other words, the force signal $F(t)$ is the following:

$$F(t) = -P_D\dot{x}(t) - P_P x(t) - \delta(t), \tag{B2}$$

where the first two terms are the feedback terms and the third term is due to an external excitation causing the system to start vibrating. Consequently, the system begins to move with displacement $x(t) = -\left(1/m2\pi\hat{f}_0\right)e^{-t/\hat{\tau}}\sin(2\pi\hat{f}_0 t)$ for $t > 0$, where $\hat{f}_0$ is the frequency of vibration with control, and $\hat{\tau}$ is the decay time with control. The velocity is as follows:

$$\dot{x}(t) = \frac{1}{m\hat{\tau}2\pi\hat{f}_0}e^{-t/\hat{\tau}}\sin(2\pi\hat{f}_0 t) - \frac{1}{m}e^{-t/\hat{\tau}}\cos(2\pi\hat{f}_0 t), \tag{B3}$$

however, the expression for the velocity can be simplified because the decay time $\hat{\tau}$ is long relative to the period of vibration. Hence, for $t > 0$,

$$\dot{x}(t) \approx -\frac{1}{m}e^{-t/\hat{\tau}}\cos(2\pi\hat{f}_0 t). \tag{B4}$$

The feedback control signal $u(t)$, from which we exclude the impulse excitation $\delta(t)$ for convenience, can be written as

$$u(t) = P_D\dot{x}(t) + P_P x(t), \tag{B5}$$

so

$$u(t) \approx -\frac{P_D}{m}e^{-t/\hat{\tau}}\cos(2\pi\hat{f}_0 t) - \frac{P_P}{m2\pi\hat{f}_0}e^{-t/\hat{\tau}}\sin(2\pi\hat{f}_0 t). \tag{B6}$$

B. Calculating the control energy

We calculate the unitless energy E of the control signal $u(t)$ where $A \triangleq -\left(P_P/m2\pi\hat{f}_0\right)$ and $B \triangleq -(P_D/m)$:

$$E = \int_0^\infty u^2(t)dt, \tag{B7}$$

so

$$E = \int_0^\infty \left(Ae^{-t/\hat{\tau}}\sin(2\pi\hat{f}_0 t) + Be^{-t/\hat{\tau}}\cos(2\pi\hat{f}_0 t)\right)^2 dt. \tag{B8}$$

E is proportional to the energy that would be dissipated in the coil of an electrodynamic actuator since the feedback control current would be proportional to $u(t)$. Multiplying out the terms we obtain the following expression for the energy:

$$E = \int_0^\infty A^2 e^{-2t/\hat{\tau}}\sin^2(2\pi\hat{f}_0 t) + B^2 e^{-2t/\hat{\tau}}\cos^2(2\pi\hat{f}_0 t) + 2ABe^{-2t/\hat{\tau}}\sin(2\pi\hat{f}_0 t)\cos(2\pi\hat{f}_0)dt. \tag{B9}$$

However, the last term under the integral can be written as $ABe^{-2t/\hat{\tau}}\sin(4\pi\hat{f}_0 t)$, which time-averages approximately to zero because the decay rate is long compared to the period of oscillation,

$$E \approx \int_0^\infty A^2 e^{-2t/\hat{\tau}}\sin^2(2\pi\hat{f}_0 t) + B^2 e^{-2t/\hat{\tau}}\cos^2(2\pi\hat{f}_0 t)dt, \tag{B10}$$

$$E \approx \frac{A^2}{2}\int_0^\infty e^{-2t/\hat{\tau}}(1 - \cos(4\pi\hat{f}_0 t))dt + \frac{B^2}{2}\int_0^\infty e^{-2t/\hat{\tau}}(1 + \cos(4\pi\hat{f}_0 t))dt. \tag{B11}$$

Equation (B11), which makes use of the power reduction law from trigonometry, can be simplified again using the fact that the decay rate is long compared to the period of oscillation, so the sinusoidal terms time-average approximately to zero,

$$E \approx \frac{A^2}{2}\int_0^\infty e^{-2t/\hat{\tau}}dt + \frac{B^2}{2}\int_0^\infty e^{-2t/\hat{\tau}}dt. \tag{B12}$$

Finally solving the integrals, we see that the control energy E_P due to changing the resonance frequency and the control energy E_D due to changing the decay time are approximately orthogonal:

$$E \approx \frac{A^2\hat{\tau}}{4} + \frac{B^2\hat{\tau}}{4} = \frac{P_P^2\hat{\tau}}{4m^2(2\pi\hat{f}_0)^2} + \frac{P_D^2\hat{\tau}}{4m^2} \overset{\Delta}{\approx} E_P + E_D. \tag{B13}$$

C. Perturbation analysis

We still need to consider how large P_P and P_D need to be in order to change the resonance frequency and the decay time. The constant $(1 + \eta_P)$ describes the increase in the resonance frequency:

$$2\pi\hat{f}_0 \approx \sqrt{\frac{k + P_P}{m}} \triangleq (1 + \eta_P)\sqrt{\frac{k}{m}} \approx (1 + \eta_P)(2\pi f_0), \tag{B14}$$

and $(1 + \eta_D)$ describes the decrease in the decay time:

$$\hat{\tau} \triangleq \frac{\tau}{1 + \eta_D}. \tag{B15}$$

Solving Eqs. (B14) and (B15) enables us to find the corresponding control gains $P_D = R\eta_D$ and $P_P = 2\eta_P k$, assuming that $\eta_P \ll 1$. Making use of Eq. (B13), we can obtain the ratio between the control energies:

$$\frac{E_P}{E_D} = \frac{1}{(2\pi\hat{f}_0)^2}\left(\frac{P_P}{P_D}\right)^2 = \frac{1}{(2\pi f_0)^2}\left(\frac{2\eta_P k}{R\eta_D}\right)^2 \approx 4Q^2\frac{\eta_P^2}{\eta_D^2}. \tag{B16}$$

We can now set the control energies equal with $E_P \overset{\Delta}{=} E_D$ and solve for η_P to obtain Eq. (13).

[1] B. Moggridge, *Designing Interactions* (MIT Press, Cambridge, MA, 2007), pp. 124–134.

[2] E. Miranda, R. Kirk, and M. Wanderley, *New Digital Musical Instruments* (A-R Editions, Middleton, WI, 2006), pp. 3–5.

[3] V. Verfaille, U. Zölzer, and D. Arfib, "Adaptive digital audio effects: A new class of sound transformations," IEEE Trans. Audio, Speech, Lang. Process. **14**(5), 1817–1831 (2006).

[4] N. Castagne, C. Cadoz, J.-L. Florens, and A. Luciani, "Haptics in computer music: A paradigm shift," in *Proceedings of EuroHaptics*, Munich, Germany (June 5–7, 2004), pp. 422–425.

[5] R. Schmidt, *Motor Control and Learning: A Behavioral Emphasis* (Human Kinetics Publishers, Champaign, IL, 1982), pp. 125–162.

[6] P. Milgram, H. Takemura, A. Utsumi, and F. Kishino, "Augmented reality: A class of displays on the reality-virtuality continuum," Proc. SPIE, **2351**, 282–292 (1994).

[7] A. Hoover, "Controls for musical instrument sustainers," U.S. Patent 6,034,316 (March 7, 2000).

[8] G. Osborne and A. Hoover, "Sustainer for a musical instrument," U.S. Patent 5,932,827 (August 3, 1999).

[9] Alternatively, v could represent the longitudinal volume velocity of air through the cross-section of a bore, and F could represent an external pressure exerted uniformly across the same cross section. In fact, the results of this paper hold as long as v and F represent any dual flow and effort acoustical system variables that are collocated.

[10] M. Puckette, T. Apel, and D. Zicarelli, "Real-time audio analysis tools for Pd and MSP," Technical Report, Ann Arbor, MI, MPublishing, University of Michigan Library, 1988 [available online: http://quod.lib.umich.edu/cgi/p/pod/dod-idx?c=icmc;idno=bbp2372.1998.315 (Last viewed September 30, 2011)].

[11] G. Franklin, J. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, 5th ed. (Prentice-Hall, Englewood Cliffs, NJ, 2005), pp. 533–543, 559–572, 236–248, 666–668, and 436–447.

[12] O. Brune, "Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency," J. Math. Phys. **10**, 191–236 (1931).

[13] J. E. Colgate and N. Hogan, "Robust control of dynamically interacting systems," Int. J. Control **48**(1), 65–88 (1988).

[14] J. Q. Sun, "Some observations on physical duality and colocation of structural control sensors and actuators," J. Sound Vib. **194**(5), 765–770 (1996).

[15] M. Balas, "Direct velocity feedback control of large space structures," J. Guid. Control **2**, 252–253 (1979).

[16] E. Berdahl, "Applications of feedback control to musical instrument design," Ph.D. dissertation, Stanford University, Stanford, CA, 2010 [available online: http://ccrma.stanford.edu/eberdahl/berdahl-thesis.pdf (Last viewed June 13, 2010)].

[17] C. Besnainou, "Modal stimulation: A sound synthesis new approach," in *Proceedings of the International Symposium on Musical Acoustics*, Dourdan, France (July 2–6, 1995), pp. 433–438.

[18] N. Fletcher and T. Rossing, *The Physics of Musical Instruments,* 2nd ed. (Springer, New York, 1998), pp. 11–13.

[19] H. Boutin and C. Besnainou, "Physical parameters of an oscillator changed by active control: Application to a xylophone bar," in *Proceedings of the 11th International Conference on Digital Audio Effects,* Espoo, Finland (September 1–4, 2008), pp. 173–176.

[20] U. Tietze, C. Schenk, and E. Gamm, *Electronic Circuits: Handbook for Design and Application*, 2nd ed. (Springer, Berlin, 2008), pp. 815–822.

[21] J. O. Smith, *Physical Audio Signal Processing: For Virtual Musical Instruments and Audio Effects* [W3K Publishing, Palo Alto, CA, 2010, available online: http://ccrma.stanford.edu/jos/pasp/ (Last viewed June, 2010)].

[22] M. Johnson and S. Elliott, "Active control of sound radiation using volume velocity cancellation," J. Acoust. Soc. Am. **98**(4), 2174–2186 (1995).

[23] L. Dozio and P. Mantegazza, "General-purpose processors for active vibro-acoustic control: Discussion and experiences," Control Eng. Pract. **15**(2), 163–176 (2007).

[24] N. Lee, E. Berdahl, G. Niemeyer, and J. Smith, III, "TFCS: Toolbox for the feedback control of sound," in *Acoustics'08: 155th Meeting of the Acoustical Society of America, 5th FORUM ACUSTICUM, and 9th Congrés Francais d'Acoustique*, Paris, France (June 30–July 4, 2008) [available online: http://ccrma.stanford.edu/eberdahl/Projects/TFCS (Last viewed June 2010)].

[25] J. O. Smith, *Introduction to Digital Filters with Audio Applications* [W3K Publishing, Palo Alto, CA, 2007, available online: http://ccrma.stanford.edu/jos/filters/ (Last viewed June 2010)].

[26] H. Fastl and E. Zwicker, *Psychoacoustics: Facts and Models*, 3rd ed. (Springer, New York, 2006), pp. 182–187.

[27] J.-J. Slotine and W. Li, *Applied Nonlinear Control* (Prentice Hall, Englewood Cliffs, NJ, 1991), pp. 126–129.

[28] M. V. Valkenburg, *Introduction to Modern Network Synthesis* (Wiley Inc., Hoboken, NJ, 1960), pp. 62–81.