Error-reducing corrections for the method of regularized Stokeslet for fluid-membrane interactions

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Abstract

We present a new high-order accuracy method for the problem of a solid object immersed in Stokes flow. The idea is to use asymptotic analysis to find the largest component of the regularization error when using the regularized Stokeslet method. Then some high-order terms are added to the largest component to make sure that the resulting velocity field is divergence free. This is not usually done by other authors.

3D Stokes Flow: Boundary Integral Formulation

The velocity at any field point x inside the Stokes flow surrounding a moving solid object can be represented by [5]

$$\mathbf{u}_i(\mathbf{x}) = -\frac{1}{8\pi\mu}\int_{\partial D} \mathbf{S}_{ij}(\mathbf{x},\mathbf{y})\,\mathbf{f}_j(\mathbf{y})\,ds(\mathbf{y}) \tag{1}$$

where

$$\mathbf{S}_{ij}(\mathbf{x},\mathbf{y}) = \frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|} + \frac{(y_i-x_i)(y_j-x_j)}{|\mathbf{x}-\mathbf{y}|^3}$$

is the singular Stokeslet corresponding to delta distribution of the force on the surface.

- Available methods, including the boundary element method (BEM) and the method of regularized Stokeslet (MRS), suffer from low accuracy at fluid locations very near the boundary.
- We focuses on improving the regularization error for the MRS while making sure that the resulting velocity field is divergence free as in the MRS. This is an important requirement.

Methodology

- According the MRS method [3, 4], the velocity of the flow at the point x can be approximated by

$$\mathbf{u}_i^\delta(\mathbf{x}) = -\frac{1}{\mu}\int_{\partial D} \mathbf{S}_{ij}^\delta(\mathbf{x},\mathbf{y})\mathbf{f}_j\,ds(\mathbf{y}) \tag{2}$$

where $\mathbf{S}_{ij}^\delta(\mathbf{x},\mathbf{y})$ is the regularized Stokeslet corresponding to the regularized delta function $\phi^\delta(\mathbf{x}-\mathbf{y})$ $(=\phi^\delta(|\mathbf{x}-\mathbf{y}|))$:

$$\mathbf{S}_{ij}^\delta(\mathbf{x},\mathbf{y}) = \frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|}H_1^\delta(|\mathbf{x}-\mathbf{y}|) + \frac{(y_i-x_i)(y_j-x_j)}{|\mathbf{x}-\mathbf{y}|^3}H_2^\delta(|\mathbf{x}-\mathbf{y}|)$$

with $(r = |\mathbf{x}-\mathbf{y}|)$

$$H_1^\delta(r) = \frac{1}{2}\int_0^r s^2\phi^\delta(s)ds - \frac{2r}{3}\int_r^\infty s\phi^\delta(s)ds + \frac{1}{6r^2}\int_0^r s^4\phi^\delta(s)ds \tag{3}$$

$$H_2^\delta(r) = \frac{1}{2}\int_0^r s^2\phi^\delta(s)ds - \frac{1}{2r^2}\int_0^r s^4\phi^\delta(s)ds \tag{4}$$

- Inspired by the work of Beale and Lai [2, 1], we use asymptotic analyses to find the largest component of the regularization error

$$\epsilon^\delta(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \mathbf{u}^\delta(\mathbf{x}) \tag{5}$$

- Two steps:
  - Find a class of regularized delta function so that the asymptotic analysis is feasible. The widely used regularized delta function introduced in [3, 4] does not belong to this class.
  - Modified the corrections to get divergence free flow field while keeping the right order of accuracy.

Main Results

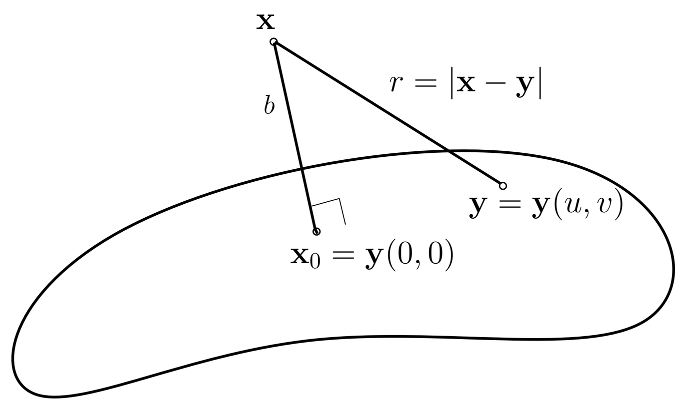
Suppose that the surface of the solid object is smooth, the force is smooth along the surface. Suppose further that $\phi^\delta(\mathbf{x})$ is such that

$$\phi^\delta(\mathbf{x}) = \frac{1}{\delta^3}\phi\left(\frac{\mathbf{x}}{\delta}\right)$$

where $\phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ is a smooth function over $\mathbb{R}^3$ satisfying the following conditions:

- the integral of ϕ over $\mathbb{R}^3$ is 1,
- the second moment of ϕ is 0,
- $|\phi(r)| \le C\cdot r^{-m}$ for $r \ge 1$ and some constant C and constant $m \ge 7$.

Let $r = |\mathbf{x}|$ and choose H_1^δ and H_2^δ as in equations (3)–(4)



Fix a point x in the flow field. Let $\mathbf{x}_0$ be a point on the surface that is closest to x and define $b = |\mathbf{x}-\mathbf{x}_0|$. Let $\mathbf{y} = \mathbf{y}(u,v)$ be a parametrization of the surface near $\mathbf{x}_0 = \mathbf{y}(0,0)$ such that $\{\mathbf{y}_u(0,0), \mathbf{y}_v(0,0)\}$ is an orthonormal set, and define $\mathbf{N} = \dfrac{\mathbf{y}_u\times\mathbf{y}_v}{|\mathbf{y}_u\times\mathbf{y}_v|}$ at $(0,0)$. We can then write $\epsilon^\delta(\mathbf{x})$ as

$$\epsilon^\delta(\mathbf{x}) = (\mathbf{f} - (\mathbf{f}\cdot\mathbf{N})\mathbf{f})\epsilon(b,\delta) + \mathcal{O}(\delta^2) \tag{6}$$

or as

$$\epsilon^\delta(\mathbf{x}) = \epsilon_1 + \epsilon_2 + \mathcal{O}(\delta^3) \tag{7}$$

with explicit expressions for $\epsilon(b,\delta)$, ϵ_1, and ϵ_2 provided for any blob ϕ^δ. The formulas (6) and (7) are uniformly with respect to x.

Remarks:

- When $b\to 0$ the second order and the third order formulas go to the same limit.
- When $b\to\infty$ both corrections go to 0 as expected.
- Formula (6) is preferred over (7) when we have limited information about the surface and the force distribution on the surface. Also, the divergence modification of (6) is much simpler.

Test Problem

For numerical test, we consider a prolate spheroid parametrized by

$$\mathbf{y} = (\cos(v)\sin(u), \sin(v)\sin(u), 2\cos(u))$$

translating with constant speed $U = 1$ along the z-axis and the regularization delta function

$$\phi^\delta(r) = \frac{1}{\delta^3}\frac{(5-2r^2/\delta^2)}{2\pi^{3/2}}e^{-r^2/\delta^2}. \tag{8}$$

Let

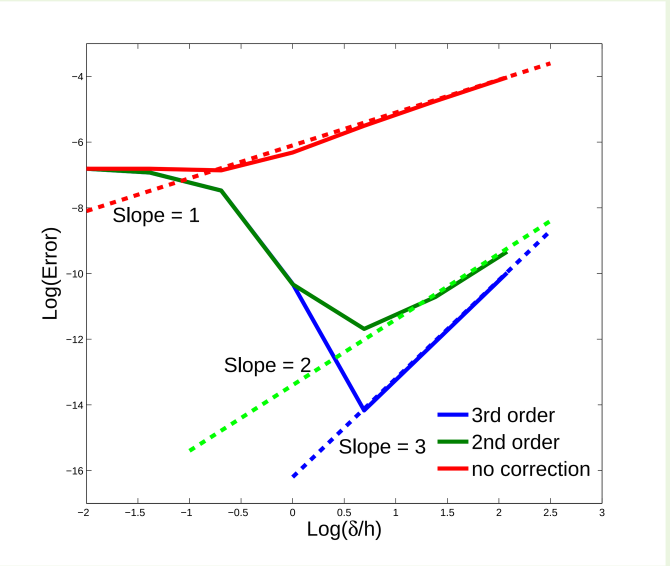
$$\mathbf{x}_0 = (\cos(v_0)\sin(u_0), \sin(v_0)\sin(u_0), 2\cos(u_0))$$

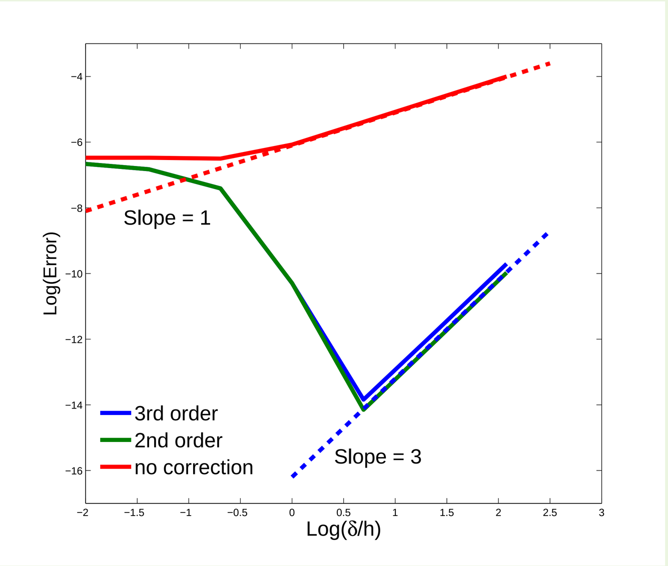
with $u_0 = v_0 = .7$ be a point on the surface and define $\mathbf{x} = \mathbf{x}_0 + b\mathbf{N}$ where N is the unit outer normal vector at $\mathbf{x}_0$. Then the distance from x to the surface is b. To evaluate the integrals, we create a regular mesh on the parameter space, uv-rectangular $[0,\pi]\times[0,2\pi]$, with step size $h = \pi/200$ in both directions, and then use a forth-order Gregory's quadrature. Thus, depending on the correction used, the total error can be written as

$$\mathcal{O}(\delta^p) + c(\delta)\mathcal{O}(h^4), \qquad p = 2,3$$

where $c(\delta)$ is a bounded function.

Velocity Errors





- The figure on the left is for $b = .001$ and on the right for $b = \pi 10^{-6} \ll h$.
- From the figures, we can see that the orders of convergence are as expected.
- Note that the figure suggests there is an optimal value of δ related to the discretization size h used for the surface integral quadrature. Below this optimal δ, we will not see the benefits of the correction because the dominant error is from the quadrature.

Approximated Divergence

3^{rd}-order-corr	2^{nd}-order-corr	without-corr	δ/h
$-3.0164e-07$	$1.6575e-09$	$3.9899e-10$	$1.6000e+01$
$-3.5252e-08$	$5.8417e-10$	$4.1460e-10$	$8.0000e+00$
$-3.2689e-09$	$2.2226e-10$	$1.8301e-10$	$4.0000e+00$
$2.3766e-10$	$1.7304e-10$	$1.5970e-10$	$2.0000e+00$
$4.4474e-10$	$2.1619e-10$	$2.0903e-10$	$1.0000e+00$
$1.4702e-10$	$-5.5023e-12$	$-1.2577e-11$	$5.0000e-01$
$1.0870e-09$	$1.0052e-09$	$6.7951e-10$	$2.5000e-01$
$3.0973e-09$	$3.1007e-09$	$7.9280e-10$	$1.2500e-01$

Future Directions

- Find an optimized relation deemed to exist between parameters (discretization size on the surface and the regularizing parameter).
- Apply the methodology to slender body theory.
- Apply the methodology to flows other than Stokes flows, for example, Brinkman flow (for porous medium), viscoelastic flows, etc.

References

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