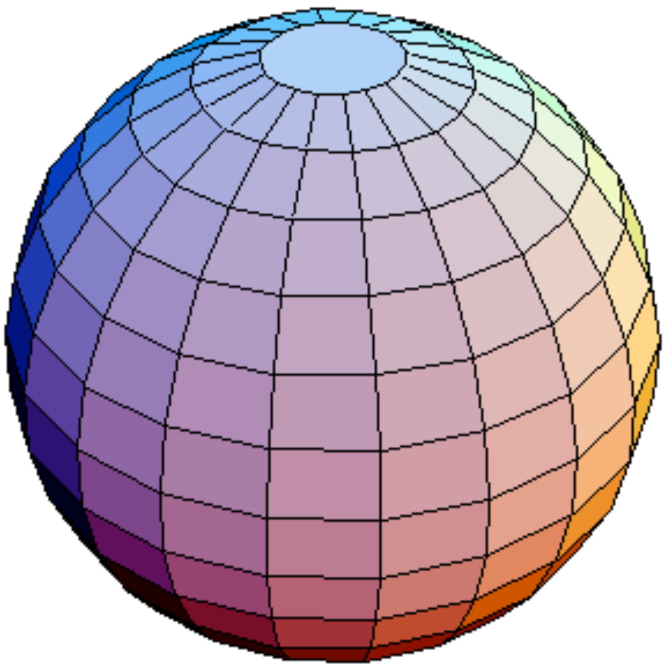
On a problem in the Stability Discussion of Rotating black holes

Irina Craciun
Research Student,
Center for Computation & Technology at LSU
irina@cct.lsu.edu

What is a metric?

- Roughly speaking, a metric is a function ds^2 describing the distance between two points in a set of events S



- Distance = the length of the shortest curve between two points
- The metric of the unit sphere

$X(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$

(arc length on a meridian):

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Einstein’s field equations

- 10 nonlinear PDEs
- Foundation of general relativity
- Describe the gravitational effects produced by a mass on the space around it (how it “curves” spacetime)
- Solutions = “metrics” of spacetime
- Minkowski metric
- Schwarzschild metric
- Kerr metric

Kerr black holes

- A more complex solution to EFE, discovered by Roy Kerr in 1963, the Kerr metric describes the geometry of spacetime around a rotating massive body
- Kerr black holes (rotating black holes) believed to be the most frequent in nature, since most stars that undergo gravitational collapse are rotating
- The ultimate question posed by the Kerr metric:

  is it stable under gravitational perturbations?
- Any body orbiting a Kerr black hole, distorting its gravitational field, is an example of a perturbation

Kerr black holes

- The stability of the Kerr metric assumed to be true; not mathematically proven
  - Physical observations of black hole behavior
  - A black hole formed in a cluster of stars eventually starves
  - No observational or numerical record of a black hole behaving otherwise

The Klein-Gordon equation on a Kerr background

- Stability under gravitational perturbations → evolution of a gravitational field propagating in the gravitational field of a Kerr black hole
- Gravitational fields - spin 2
- Scalar fields - simplest fields, spin 0
- Current studies focused on the case of the scalar field
- The (reduced) Klein-Gordon equation on a Kerr background
- Describes a spin-0 quantum field (pion field) propagating in the gravitational field of a rotating black hole
- Stability of the Kerr metric ⇔ non-exponentially growing solutions

$$(\Box + \mu^2)u = 0$$

The Klein-Gordon equation on a Kerr background

- Pions ← the Klein-Gordon field ← dictated by the black hole's gravitational field ↔ the coefficients of the □ - operator

The □ - operator

$$\Box = \frac{\gamma}{\Delta\Sigma}\partial_t^2 + \frac{4imMar}{\Delta\Sigma}\partial_t - \frac{1}{\Sigma}\partial_r \Delta \partial_r - \frac{1}{\Sigma \sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\Sigma}\left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\gamma = \Sigma(r^2 + a^2) + 2Mra^2\sin^2\theta$$

- t - time
- r - radial coordinate
- θ - angular coordinate
- μ - mass of solution field
- m - angular momentum # of the solution
- M - mass of black hole
- i - imaginary unit
- a - rotational parameter

Multiplication of Operators

$$\frac{\partial}{\partial x}\left(xy^2\frac{\partial}{\partial y}\right)=y^2\frac{\partial}{\partial y}+xy^2\frac{\partial^2}{\partial x\partial y}$$

$$O_1=xy\frac{\partial}{\partial x}$$

$$O_2=xy^2\frac{\partial}{\partial y}$$

$$O_1O_2=xy^3\frac{\partial}{\partial y}+x^2y^3\frac{\partial^2}{\partial x\partial y}$$

$$O_2O_1=xy^3\frac{\partial}{\partial x}+x^2y^3\frac{\partial^2}{\partial x\partial y}$$

Carter and Lenaghan's symmetry operator

$$\bar{\Box} = \frac{1}{\Sigma}\left\{a^2\left[\cos^2\theta\frac{(r^2+a^2)^2}{\Delta}+r^2\sin^2\theta\right]\partial_t^2+2ima\left(r^2+a^2\cos^2\theta\frac{r^2+a^2}{\Delta}\right)\partial_t-m^2\left(\frac{a^4\cos^2\theta}{\Delta}+\frac{r^2}{\sin^2\theta}\right)-a^2\cos^2\theta\partial_r\Delta\partial_r+\frac{r^2}{\sin\theta}\partial_\theta\sin\theta\partial_\theta\right\}$$

- Carter and Lenaghan's result (1979)
- Commutes with $\Box$
- Presence of 2nd order time derivatives in the $\Box$ operator poses a structural problem; natural ideas which emerge:
  - Find a symmetry (commuting?) operator which only contains first order time derivatives
  - Use $\Box$ to find this new operator → replace the 2nd-rder time derivative in the $\Box$ operator with the expression for this derivative obtained from the Klein-Gordon equation

Results

$$\hat{\square} = \frac{2ima(r^2 + a^2)\Sigma}{\gamma}\partial_t + \frac{a^2\Delta\sin^2\theta}{\gamma}\partial_r\Delta\partial_r + \frac{(r^2 + a^2)^2}{\gamma\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{m^2\left[a^4\sin^4\theta - (r^2 + a^2)\right]}{\gamma\sin^2\theta}$$

$$\hat{\square}\square - \square\hat{\square} = A\square + B\partial_r\square + C\partial_\theta\square$$

Results

$$C = \frac{8Mra^2(r^2+a^2)^3 \sin\theta\cos\theta}{\Sigma\gamma^2}$$

$$B = \frac{-4M\Delta a^2 \sin^2\theta\left[2r^2a^2\Delta\sin^2\theta + \Sigma(r^4-a^4)\right]}{\Sigma\gamma^2}$$

$$A = \frac{1}{\gamma^3\Sigma}\left\{2Ma^2\sin^2\theta\left(T_1T_2 - \gamma\Delta T_3\right) + 4Mra^2(r^2+a^2)^3\left[(2\cos^2\theta - \sin^2\theta)\gamma + 4\Delta a^2\sin^2\theta\cos^2\theta\right]\right\}$$

$$T_1 = 2r^2a^2\Delta\sin^2\theta + \Sigma(r^4-a^4)$$

$$T_2 = (2r-2M)\gamma + 4r\Delta(r^2+a^2) - 4M(r^4-a^4)$$

$$T_3 = 4ra^2\Delta\sin^2\theta + 2r^2a^2(2r-2M)\sin^2\theta + 2r(r^4-a^4) + 4r^3\Sigma$$

Additional Results

- Initial data → numerically → solution to the Klein-Gordon equation
- $\bar{\Box}$ → constraint → conserved quantity
- Currently no interpretation (angular momentum?)

$$\Box' = a^2 \sin^2 \theta \partial_t^2 + 2ima\partial_t + \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta - \frac{m^2}{\sin^2\theta}$$

$$\boxed{\Box'\Box - \Box\Box' = A'(\Box) + B'(\partial_\theta \Box)}$$

$$A' = \frac{2a^2(2\cos^2\theta - \sin^2\theta)}{\Sigma}$$

$$B' = \frac{4a^2 \sin\theta\cos\theta}{\Sigma}$$

Broader Impact

- Numerical simulations of collisions between two black holes
  - assume the Kerr metric to be stable
  - show no instability
  - if it were to be proven that the Kerr metric is unstable - why do these simulations show no sign of instability?
    - Numerical dissipation in a finite difference approximation
    - Relatively short evolution times of typical numerical relativity codes
    - Recently improved numerical relativity codes are able to evolve multiple orbits of black hole binaries
    - Equations considered in this theoretical analysis are the EFE *liniarized* about the Kerr black hole; numerical simulations solve the full *nonlinear* equations

Broader Impact - LIGO

- LIGO (Laser Interferometer Gravitational-Wave Observatory)
  - Detection of cosmic gravitational waves and their scientific study
  - Strongest sources of such waves - black hole collisions
  - Towards the end phase of the collision - solutions approximated by small perturbations of the Kerr metric (extraction of wave signals from the end phase of a black hole collision)
  - Studies of stellar populations → estimates of the expected number, frequency and location of black hole collision events in space
  - Einstein's theory → limits on the amplitude of gravitational waves detected by LIGO
  - No unexpected events in detected data

Acknowledgements

- The author wishes to express her appreciation and gratitude to **Dr. Horst Beyer**, **Dr. Ed Seidel**, **Dr. Gabrielle Allen**, **Dr. Peter Diener**, and the entire scientific community of the Center for Computation & Technology at Louisiana State University, for providing the opportunity, guidance and means for the author's research.
- The author also wishes to express her gratitude to Louisiana State University's Mathematics Department, and **Dr. Larry Smolinsky** for helping support this research.