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On a Problem in the Stability Discussion of Rotating Black Holes

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Abstract

When a sufficiently massive rotating star collapses its core keeps rotating, passing this motion on to the resulting black hole through conservation of angular momentum. We study the evolution of 0-spin particles (described by the Klein-Gordon equation) in the gravitational field of a stationary rotating black hole (Kerr black hole). An important open problem is the existence of exponentially growing solutions of the Klein-Gordon equation. Or more generally: Is the Kerr gravitational field stable against small perturbations? This problem has important implications in general relativity, particularly in the numerical treatment of the collision of two Kerr black holes, and the problem of extracting a physical signal from the far zones of the gravitational field after the collision. This problem is of great interest to gravitational wave experimentalists working with the LIGO detector.

The Kerr gravitational field has generalized symmetries (in addition to stationarity and axial symmetry), which leads to a wave-type symmetry operator that commutes with the Klein-Gordon operator. However, the occurrence of a second order time derivative in this operator creates technical complications in a mathematical setting; trying to avoid this problem, we have explored the results of substituting the Klein-Gordon equation into the symmetry operator. This led to another potential symmetry operator, containing only a first order time derivative. The commutator between this new operator and the Klein-Gordon operator was calculated, and was discovered to have a structure previously found for a class of symmetry operators already considered in the literature. This is expected to have important implications in the discussion of the stability of the solutions to the Klein-Gordon equation.

1. Introduction

1.2 Black holes

The foundation of general relativity are Einstein's field equations, a collection of 16 nonlinear partial differential equations which describe the gravitational effects produced by a body's mass on the space around it – or, the way this mass "curves" spacetime. Solutions to Einstein's field equations are "metrics" of spacetime; a metric is a function, g, describing the distance between two points in a set S, which has the following properties for any x and y in S:

$$\begin{aligned}
&g(x, y) \geq 0; \\
&g(x, y) + g(y, z) \geq g(x, z); \\
&g(x, y) = g(y, x); \\
&g(x, y) = 0 \Leftrightarrow x = y; \qquad (1)
\end{aligned}$$

The simplest example of a metric, and also a trivial solution to Einstein's field equations, is the fairly intuitive Minkowski metric, where the three ordinary space coordinates and one time coordinate describe a spacetime:

$$ds^2 = -c^2dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{dl^2} \tag{2}$$

As an example, for $ds^2=0$, we have $c^2dt^2 = dl^2$, or $dv^2 = c^2$, or an object moving with the speed of light.

Karl Schwarzschild discovered the simplest non-trivial exact solution to Einstein's field equations in 1916, only months after their publication. This solution is known as "the Schwarzschild metric":

$$ds^2 = -c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\Omega^2 = g_{ij}dx^idx^j \tag{3}$$

where G is the gravitational constant, and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{4}$$

is the standard metric of a sphere, where we can recognize the spherical coordinates. This metric describes the geometry of spacetime around a spherical, *non-rotating* body of mass M, and is a reasonable approximation to slow-rotating bodies, such as the Sun or the Earth. A crucial element to the Schwarzschild metric is the "Schwarzschild radius", r_s, proportional to the mass of the object (a property characteristic to every mass M). r_s is one of the singularities of the Schwarzschild metric, or the points where the solution blows up; the other singularity is $r = 0$. As one would expect, the solution is stable for any radius larger than the radius of the gravitating body, R; so, as long as $R > r_s$ there is no concern about the validity of the solution. However, for $R < r_s$, the solution is still stable, but it then describes a black hole.

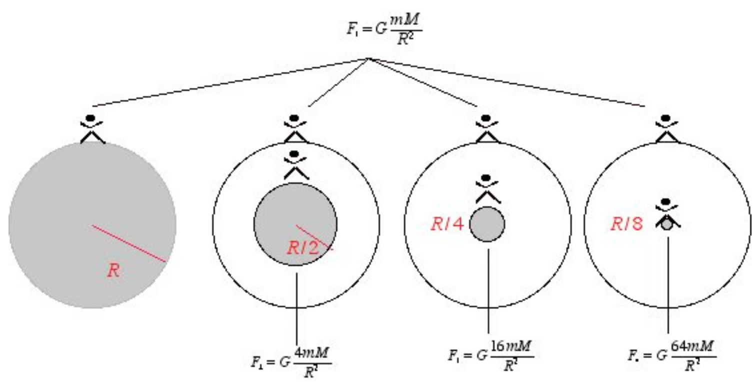

Figure 1. Black hole analogy

The surface $r = r_s$ is called the "event horizon" of the black hole, and it delimitates the surface past which light can no longer escape the black hole's gravitational field. In other words, if any non-rotating body modifies its radius to the point where it becomes less than the Schwarzschild radius corresponding to the object's mass, it becomes a Schwarzschild black hole (a non-rotating black hole).

It is easiest to imagine this using the following well-known analogy [1]: suppose the Earth has shrunk to half its radius, but kept its mass, M, and is now twice as dense (figure 1). At distance R, where the old surface used to be, the Earth's gravitational pull on your body remains unchanged; on the Earth's new surface however, the gravitational force is four times F_1. If the Earth shrinks again to half its radius, the gravitational force on the new surface is sixteen times F_1. Imagine this process continuing until the whole Earth's mass is condensed in a sphere the size of a cherry: the gravitational force on this surface is so strong, that the escape speed exceeds the speed of light. The Earth has now become a black hole.

1.3. Kerr black holes and the Kerr metric

A more complex exact solution of Einstein's field equations was discovered by Roy Kerr in 1963; the Kerr metric describes the geometry of spacetime around a *rotating* massive body, and Kerr black holes (rotating black holes). These are thought to be the most frequent in nature, since most stars that undergo gravitational collapse are rotating. The ultimate question posed by the Kerr metric is whether or not it is stable under gravitational perturbations. A perturbation is "almost" a Kerr solution, but not quite. A simpler example is that of a perturbation of the Schwarzschild metric: instead of the initial $ds^2 = g_{ij}dx^i dx^j$, a perturbation would be:

$$ds^2 = (g_{ij} + \delta g_{ij})dx^i dx^j. \tag{5}$$

In this sense, the metric is stable if, by replacing the new ds^2 into Einstein's field equations, all the terms containing δg_{ij} will cancel each other out, and the final result will be the same as if we hadn't considered the perturbation.

For a rotating black hole, a perturbation is any body orbiting it and distorting its gravitational field. The stability of the Kerr metric is assumed to be true, although it has never been mathematically proved. This assumption is based upon physical observations of black hole behavior. One of the most relevant examples is that of a black hole that is formed in a cluster of stars; the newly formed black hole "swallows" the stars in its vicinity, but it eventually starves and remains a simple rotating black hole – a Kerr solution which has stabilized under perturbations. There is no record of a black hole behaving otherwise, and numerical simulations of black holes also show no sign of instability of the Kerr metric.

2. The Klein-Gordon Equation on a Kerr Background

In order to answer the question of the stability of the Kerr metric under gravitational perturbations, we would have to study the evolution of a gravitational field propagating in the gravitational field of a rotating black hole. However, gravitational fields have spin 2, while the simplest kind of field, the scalar field, has spin 0. Current studies are focused on the simplest case of the scalar field in hope that the proof of the stability of the Kerr metric for this case will inspire the proof of stability under the more complex influence of a gravitational field.

We are studying equation (6), the Klein-Gordon equation on a Kerr background, describing a spin-0 quantum field (such as a pion field) propagating in the gravitational field of a rotating black hole:

$$(\square + \mu^2)u = 0 \tag{6}$$

where the unknown, u, is the above-mentioned quantum field, and:

$$\square = \frac{\gamma}{\Delta\Sigma}\partial_t^2 + \frac{4imMar}{\Delta\Sigma}\partial_t - \frac{1}{\Sigma}\partial_r \Delta \partial_r - \frac{1}{\Sigma\sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\Sigma}\left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right) \tag{7}$$

$$\Delta = r^2 + a^2 - 2Mr \tag{8}$$

$$\Sigma = r^2 + a^2\cos^2\theta \tag{9}$$

$$\gamma = \Sigma(r^2 + a^2) + 2Mra^2\sin^2\theta = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \tag{10}$$

Both **Δ** and **Σ** are common operators of the Kerr metric; **γ** is an additional operator introduced by the author, for the convenience of the notation. The variables in this equation are **t** (time), **r** (radial coordinate roughly measuring the distance from the black hole), and **θ** (angular coordinate). All the other letters determine constants: **μ** (mass of the solution field), **m** (angular momentum quantum number of the solution, an integer), **M** (mass of the black hole), **i** (the irrational number) and **a** (rotational parameter of the black hole, a measure of spin; for a = 0 we have a Schwarzchild, or non-rotating black hole). In quantum theory, particles are described by fields; in this case, scalar particles like pions (subatomic particles with 0-spin) are described by the Klein-Gordon field; the latter's evolution is in turn "dictated" by the black hole's gravitational field, which materializes in the coefficients of the $\Box$-operator, derived from the Kerr metric coefficients. The open question pertaining to equation (6) is proving the stability of its solutions, or proving that its solutions are not exponentially growing.

3. Methodology

The author's research was based upon Carter and Lenaghan's result [2], a symmetry operator which commutes with $\Box$:

$$\bar{\Box} = \frac{1}{\Sigma}\left\{a^2\left[\cos^2\theta\frac{(r^2+a^2)^2}{\Delta}+r^2\sin^2\theta\right]\partial_t^2+2ima\left(r^2+a^2\cos^2\theta\frac{r^2+a^2}{\Delta}\right)\partial_t-m^2\left(\frac{a^4\cos^2\theta}{\Delta}+\frac{r^2}{\sin^2\theta}\right)-a^2\cos^2\theta\partial_r\Delta\partial_r+\frac{r^2}{\sin\theta}\partial_\theta\sin\theta\partial_\theta\right\} \tag{11}$$

The presence of second-order time derivatives in the $\Box$-operator poses a structural obstacle in the function analysis treatment of the problem of the stability of the solutions of equation (6), since it is a case not yet treated in literature. The natural idea that emerges is to find an additional symmetry operator which would only contain a first-order time derivative. Ideally, this operator would also commute with $\Box$-operator, this being the author's initial intention. Since a symmetry operator, by definition, maps solutions into solutions, another natural idea is to use $\Box$, present in the actual equation, to derive the new operator. The method used was replacement of the second-order time derivative in the $\Box$-operator, with the expression for this derivative obtained from equation (6):

$$(1) \Rightarrow \partial_t^2 u = -\frac{\Delta\Sigma}{\gamma}\left[\frac{4imMar}{\Delta\Sigma}\partial_t-\frac{1}{\Sigma}\partial_r\Delta\partial_r-\frac{1}{\Sigma\sin\theta}\partial_\theta\sin\theta\partial_\theta-\frac{m^2}{\Sigma}\left(\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right)+\mu^2\right]u \tag{12}$$

This expression is now introduced in (11); after some calculations of the coefficients, we obtain the new symmetry operator:

$$\hat{\Box}_1 = \frac{2ima(r^2+a^2)\Sigma}{\gamma}\partial_t+\frac{a^2\Delta\sin^2\theta}{\gamma}\partial_r\Delta\partial_r+\frac{(r^2+a^2)^2}{\gamma\sin\theta}\partial_\theta\sin\theta\partial_\theta+\frac{m^2\left[a^4\sin^4\theta-(r^2+a^2)\right]}{\gamma\sin^2\theta}$$
$$-a^2\mu^2\frac{\cos^2\theta(r^2+a^2)^2+\Delta r^2\sin^2\theta}{\gamma} \tag{13}$$

The expectations for this operator to commute with $\Box$ were not fulfilled. After a simple Maple check, the commutator between the two, $\hat{\Box}_1\Box - \Box\hat{\Box}_1$, was not equal to 0. Unfortunately, Maple did not allow for a structure of this commutator to be easily noticed, which is why the investigation of this operator and its relationship to $\Box$ had to be done mostly by hand. The amount of calculation required to obtain all the terms of this commutator without the aid of a computer would be close to impossible, and prone to human error; therefore, the author attempted calculation of a few select, simpler terms, in hope that this would lead to a reasonable "guess" of the commutator's structure, which could later be validated by Maple. Only a few of these trial calculations will be presented here, for the purpose of example.

The terms containing the θ-derivatives were easiest to calculate. The coefficient of $\partial_t\partial_\theta$ in the commutator is:

$$C_1 = \frac{2ima(r^2+a^2)}{\Sigma\gamma}\frac{16a^2M^2r^2(r^2+a^2)^2\sin\theta\cos\theta}{\Delta\Sigma\gamma} \tag{14}$$

The coefficient of ∂_t in $\Box$, $\frac{4imMar}{\Delta\Sigma}$, remains unchanged if differentiated after θ, and is also present in C_1:

$$C_1 = C\frac{4imMar}{\Delta\Sigma}\partial_t\partial_\theta \tag{15}$$

where

$$C = \frac{8Mra^2(r^2+a^2)^3\sin\theta\cos\theta}{\Sigma\gamma^2} \tag{16}$$

Similarly, the coefficient of ∂_θ^3 in the commutator was calculated to be

$$C_2 = -\frac{8Mra^2(r^2+a^2)^3\sin\theta\cos\theta}{\gamma^2\Sigma^2} \tag{17}$$

This is also a multiple of the coefficient of ∂_θ^3 in the differentiation by θ of $\Box$:

$$C_2 = C\left(-\frac{1}{\Sigma}\right) \tag{18}$$

where C is the same as defined above in (16).

These and other similar "coincidences" suggest the possibility that $C(\partial_\theta\Box)$ could be found in the commutator. Further investigation of the coefficients containing ∂_θ in the commutator, and comparison to the corresponding coefficients of $\partial_\theta\Box$ proved that indeed this is the case.

It is obvious that the absence of t in the coefficients leads to all fourth and third degree time derivatives to cancel each other out in the commutator. This led to the conclusion that $\partial_t\Box$ should not be expected to appear in the commutator, because it would imply the existence of third-order time derivatives. Simple calculations showed that no fourth-degree derivatives appear in the commutator, which suggested that only first-order derivatives of $\Box$ should be searched for in the structure of the commutator.

The r-derivatives proved to be the most difficult to compute. A fairly simple calculation is that of the coefficient of $\partial_r\partial_t^2$ in the commutator:

$$C_3 = -\frac{4Ma^2\sin^2\theta}{\Sigma}\frac{2r^2a^2\Delta\sin^2\theta+\Sigma(r^4-a^4)}{\Sigma\gamma} \tag{19}$$

Comparison of C_3 with the coefficient of $\partial_r\partial_t^2$ in $\partial_r\Box$ showed that the first is a multiple of the latter, by the factor B:

$$B = \frac{-4M\Delta a^2\sin^2\theta\left[2r^2a^2\Delta\sin^2\theta+\Sigma(r^4-a^4)\right]}{\Sigma\gamma^2} \tag{20}$$

This led to the idea that $B(\partial_r\Box)$ could potentially be found in the commutator. The same term B was also obtained from the comparison between the coefficients of $\partial_r\partial_\theta^2$ in the commutator (C_4) and in $\partial_r\Box$; once again, the first is a multiple of the latter by the factor B:

$$C_4 = \frac{4M\Delta a^2\sin^2\theta\left[2r^2a^2\Delta\sin^2\theta+\Sigma(r^2+a^2)(r^2-a^2)\right]}{\Sigma^2\gamma^2} = B\left(-\frac{1}{\Sigma}\right) \tag{21}$$

The rest of the calculations pertaining to B were left for the Maple check.

After the full calculation of the coefficient of ∂_t^2 in the commutator, the coefficients of ∂_t^2 in $\partial_r\Box$ and $\partial_\theta\Box$ were both present; however, it was clear that an additional multiple of $\Box$ had to be present, this time a multiple containing no derivatives: $A\Box$. This coefficient proved hardest to calculate, but was found to be:

$$A = \frac{1}{\gamma^3\Sigma}\left\{2Ma^2\sin^2\theta\left(T_1T_2 - \gamma\Delta T_3\right) + 4Mra^2(r^2+a^2)^3\left[(2\cos^2\theta - \sin^2\theta)\gamma + 4\Delta a^2\sin^2\theta\cos^2\theta\right]\right\} \tag{22}$$

where:

$$T_1 = 2r^2a^2\Delta\sin^2\theta + \Sigma(r^4 - a^4) \tag{23}$$

$$T_2 = (2r - 2M)\gamma + 4r\Delta(r^2 + a^2) - 4M(r^4 - a^4) \tag{24}$$

$$T_3 = 4ra^2\Delta\sin^2\theta + 2r^2a^2(2r - 2M)\sin^2\theta + 2r(r^4 - a^4) + 4r^3\Sigma \tag{25}$$

4. Results

No additional terms seemed to emerge from the selected calculations, so the following hypothesis was subjected to a Maple verification:

$$(\hat{\Box}_1\Box - \Box\hat{\Box}_1)u = (\mathrm{A}\Box + \mathrm{B}\partial_{\mathrm{r}}\Box + \mathrm{C}\partial_\theta\Box)u \tag{26}$$

where A, B, and C were defined earlier in (22), (20), and (16) respectively. A simple Maple check showed that

$$\hat{\Box}_1\Box - \Box\hat{\Box}_1 - (\mathrm{A}\Box + \mathrm{B}\partial_{\mathrm{r}}\Box + \mathrm{C}\partial_\theta\Box) \neq 0 \tag{27}$$

but also that it contained only multiples of μ; this suggested investigation of the 0-mass case of the Klein-Gordon equation, or $\Box u = 0$. Considering this case instead of the initial equation (6), the final $\hat{\Box}$ operator was obtained: identical to the previous $\hat{\Box}_1$ except for the last term containing μ:

$$\hat{\Box} = \frac{2ima(r^2+a^2)\Sigma}{\gamma}\partial_t + \frac{a^2\Delta\sin^2\theta}{\gamma}\partial_r\Delta\partial_r + \frac{(r^2+a^2)^2}{\gamma\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{m^2\left[a^4\sin^4\theta - (r^2+a^2)\right]}{\gamma\sin^2\theta} \tag{28}$$

A simple Maple check proved that this symmetry operator has the following property:

$$\hat{\Box}\Box - \Box\hat{\Box} = \mathrm{A}\Box + \mathrm{B}\partial_{\mathrm{r}}\Box + \mathrm{C}\partial_\theta\Box \tag{29}$$

Although it doesn't commute with $\Box$, the new symmetry operator defined in (28) does have this interesting property; similar such symmetry operators having this property have been studied in literature [3]. Further investigation of this operator could prove to be helpful in deciding the stability of the solutions of the Klein-Gordon equation on a Kerr background.

For a set of initial data, we obtain numerically a unique solution to equation (6). The symmetry operator $\bar{\Box}$ reveals a constraint on each of these solutions; by Noethe's theorem, this constraint induces a conserved quantity, or a quantity defined in terms of the field, which does not change in time (like the field's energy). Currently there is no interpretation for this quantity, but it is believed to be related to the angular momentum of the field. This quantity is being investigated because it would be the simplest way to decide the stability of the solutions of equation (6). Similarly to the method used to obtain $\hat{\Box}$, another symmetry operator was calculated, this time by eliminating the r-derivatives:

$$\square' = a^2 \sin^2 \theta \partial_t^2 + 2ima\partial_t + \frac{1}{\sin\theta} \partial_\theta \sin\theta \partial_\theta - \frac{m^2}{\sin^2\theta} \tag{30}$$

This operator has the following property:

$$\square'\square - \square\square' = A'(\square) + B'(\partial_\theta \square) \tag{31}$$

where

$$A' = \frac{2a^2(2\cos^2\theta - \sin^2\theta)}{\Sigma} \tag{32}$$

$$B' = \frac{4a^2 \sin\theta \cos\theta}{\Sigma} \tag{33}$$

It is not yet clear whether or not this latter operator will prove to be of any importance, but it could be of interest to the investigation of the conserved quantity; the fact that in operator (30) all the coefficients of the derivatives are positive could play a part in the proof that the conserved quantity is also positive, a property which would be useful in deciding stability.

5. Broader Impact

The stability of the Kerr metric is already assumed to be true in numerical simulations of the collision between two Kerr black holes. These simulations show the black holes merging, and finally stabilizing into one Kerr black hole; no instability seems to be present. If it were to be proved that the Kerr metric is not stable under small perturbations, the entire theory behind these numerical simulations would have to be reconsidered, and an answer will need to be found to the question of why the simulations show no sign of instability. Most probably, this would be the cause of numerical dissipation in the code: finite difference could make it impossible to observe small-scale features. A mathematical proof of the stability of the Kerr metric under small perturbations is in great demand, because it would give a true physical relevance to these simulations.

The LIGO project (The Laser Interferometer Gravitational-Wave Observatory) is dedicated to the detection of cosmic gravitational waves and their scientific study. Perhaps the strongest sources of such gravitational waves are black hole collisions, the most spectacular cosmic events. Towards the end zones of the collision, the solutions are approximated to small perturbations of the Kerr metric; therefore, the stability of the Kerr metric is also important for the LIGO detector, and the extraction of wave signals from these far end zones of a black hole collision.

The question remains whether $\bar{\square}$ and other symmetry operators derived from it could be used to decide the stability of the Kerr metric. The most extensive research in this area is being conducted by B. Carter and R.G. McLenaghan, M. Willard and H.R. Beyer. Differential geometry methods which some employ are not best suited to attack such a stability question, a problem that will most likely find its solution in function analysis. Hopefully, further study of the new symmetry operators presented will play a part in the function analysis treatment of the stability of the solutions of the Klein-Gordon equation on a Kerr background.

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