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RESULTS ON THE DIFFUSION EQUATION WITH ROUGH COEFFICIENTS*

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Abstract. We study the behavior of the solutions of the stationary diffusion equation as a function of a possibly rough (L^∞-) diffusivity. This includes the boundary behavior of the solution maps, associating to each diffusivity the solution corresponding to some fixed source function, when the diffusivity approaches infinite values in parts of the medium. In n-dimensions, $n \geqslant 1$, by assuming a weak notion of convergence on the set of diffusivities, we prove the strong sequential continuity of the solution maps. In one dimension, we prove a stronger result, i.e., the unique extendability of the map of solution operators, associating to each diffusivity the corresponding solution operator, to a sequentially continuous map in the operator norm on a set containing "diffusivities" assuming infinite values in parts of the medium. In this case, we also give explicit estimates on the convergence behavior of the map. In the end, we provide connections to preconditioning.

Key words. diffusion equation, diffusion operator, rough coefficients, preconditioning, first-order formulation, mixed formulation, dependence on diffusivity

AMS subject classifications. 35J25, 47F05, 65J10, 65N99

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1. Motivation.

Numerical methods for the diffusion equation with rough coefficients have been studied extensively [6, 7, 8, 16, 17, 18, 20] in the preconditioning (multigrid, domain decomposition, and related iterative methods) literature starting in the early 1980s and still continue to be an active area of research in various preconditioning efforts [26, 27]. This article came about out of a need for deeper understanding of the performance of preconditioners and their connection to the underlying PDE.

In a recent article [3], the first author constructed a new preconditioning strategy with rigorous justification which is comparable to algebraic multigrid. It is shown in [3, 4, 5] that analytical tools such as singular perturbation analysis give valuable insight about the asymptotic behavior of the solution of the underlying PDE; hence, providing feedback for preconditioner construction.

According to experience, the performance of a preconditioner depends essentially on the degree to which the preconditioner operator approximates the underlying operator. Then, the fundamental need is to explain the effectiveness of the preconditioner and to justify that rigorously. In that respect, one can view the tools in this article as steps towards adding tools to the arsenal of methods of analysis for rigorous justification at the interface of preconditioning and operator theory. An alternative approach using unbounded operators was taken by Faber, Manteuffel, and Parter [14]. We provide connections from the results here to preconditioning in the concluding remarks. Further connections will be the subject of future research.

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2. Introduction. The diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(\, p \operatorname{grad} u) + f \tag{2.1}$$

describes general diffusion processes, including the propagation of heat, and flows through porous media. Here u is the density of the diffusing material, p is the diffusivity of the material, and the function f describes the distribution of *sources* and *sinks*. This paper focuses on stationary solutions of (2.1) satisfying

$$-\operatorname{div}(\, p \operatorname{grad} u) = f \ . \tag{2.2}$$

For instance, the fictitious domain method and composite materials are sources of rough coefficients; see the references in [18]. Important current applications deal with composite materials whose components have nearly constant diffusivity, but vary by several orders of magnitude. In composite material applications, it is quite common to idealize the diffusivity by a piecewise constant function and also to consider limits where the values of that function approach zero or infinity in parts of the material.

Results of such study were given first by Lions [19]. In his lecture notes, he considers the limit of the solution of (2.2) where the limit is associated to a one-parameter family of piecewise constant diffusivities approaching zero on a subdomain. The same piecewise constant one-parametric approach was used in [7, 17], but with diffusivities approaching infinity on a subdomain. The limitation of the one-parametric approach is its dependence on the particular approximating sequence. To the best knowledge of the authors, this paper is the first to address these questions in the necessary generality. Hence, we consider general families of diffusivities that are not necessarily piecewise constant. In addition, due to the atomistic structure of matter, the physical treatment of diffusion involves regular (C^{∞}-) diffusivity. It is unclear to what extent the idealization of diffusivity by piecewise constant coefficients has the capability to capture the underlying physics. Mathematically, the severe contrast in diffusivity should be represented by a regular function whose size is changing drastically over small distances in interface regions. In this paper, we demonstrate that the assumption of piecewise constant diffusivities is meaningful by showing a continuous dependence of the solutions on the diffusivity.

Furthermore, the diffusion equation is meaningless if the *diffusivity* is infinite or zero in regions of the material. Physics requires nowhere vanishing diffusivity in the interior of the material. As a consequence, only the relative size of diffusivities should be significant. Therefore, physically, one might expect that both types of the above limits are equivalent, but mathematically there are differences. The limit of the solution as diffusivity approaches infinite values exists. However, only the limit of the scaled solution exists as diffusivity approaches zero values (see Example 2 for both cases). That is why we choose to work with diffusivity approaching infinity. We will refer to these cases as *asymptotic cases*.

Also, the treatment in [7, 17] considers only limits on specific parts of the material. In this connection, it should also be remarked that, although (2.1), (2.2) are linear equations, in general, their solutions depend nonlinearly on the coefficients.

For the treatment of these questions, we use methods from operator theory. For this, we use a common approach to give (2.1) a well-defined meaning that, in a first step, represents the diffusion operator

$$-\operatorname{div} p \operatorname{grad} \tag{2.3}$$

as a densely defined, linear, and positive self-adjoint operator A_p in a suitable Hilbert space. As a result, (2.2) is represented by

$$A_p u = f \ ,$$

where f is an element of the Hilbert space, and u is from the domain, $D(A_p)$, of A_p.[1]

Specifically, we treat the class $\mathcal{L}$ of diffusivities $p \in L^\infty(\Omega)$ that are almost everywhere (a.e.) $\geqslant \varepsilon$ on Ω for some $\varepsilon > 0$, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}^*$, is some nonempty open subset. Using Dirichlet boundary conditions, A_p defines an operator in the complex Hilbert space $L^2_{\mathbb{C}}(\Omega)$. For nonsmooth p, the domain of A_p depends heavily on p. This fact significantly complicates the study of sequences of functions of A_p.

In this paper, we turn to a first-order formulation of (2.2) which is often referred to as mixed formulation in the discretization literature [10]. The first-order formulation was popularized in the least squares finite element community by the so-called FOSLS pioneering paper [12]. Here, we provide the self-adjointness of a corresponding operator $\hat{A}_{1/p}$ in a Hilbert space. The key property of $\hat{A}_{1/p}$ is that its domain, $D(\hat{A}_{1/p})$, is independent of p. This property is exploited in establishing the continuity of the solutions $A_p^{-1} f$ as a function of p. Moreover, $\hat{A}_{1/p}$ remains defined for the asymptotic cases when (2.1), (2.2) are ill-defined. This fact is used in the study of the asymptotic cases.

Specifically, for $p \in \mathcal{L}$ and by assuming a weak notion of convergence in $\mathcal{L}$, we show that the maps that associate p to the operator A_p^{-1} and $-p \nabla A_p^{-1}$, respectively, are strongly sequentially continuous; see Theorem 5.6 and Corollary 5.7. In particular, this shows in these cases that the approximation by discontinuous coefficients to physical diffusivity is indeed meaningful. In addition, for the case $n = 1$ and bounded open intervals of $\mathbb{R}$, we show stronger results that include also the asymptotic cases, except where the asymptotic *diffusivity* is a.e. infinite on the interval. In this case, the maps that associate ϖ to the operator $A_{1/\varpi}^{-1}$ and $-(1/\varpi) \nabla A_{1/\varpi}^{-1}$, respectively, have unique extensions to sequentially continuous maps in the operator norm on the set of a.e. positive elements of $L^\infty(\Omega) \backslash \{0\}$; see Corollaries 6.3 and 6.4. In addition, an explicit estimate of the convergence behavior of the maps is given; see Theorem 6.2. It is still an open problem whether the last results are generalizable to dimensions $n \geqslant 2$.

3. Basic notation. Mainly, this section introduces basic notation. In particular, an operator theoretic definition of Sobolev spaces is given that is based on weak derivative operators, instead of distributions. In such formulation, the completeness of the Sobolev spaces is an obvious consequence of the closedness of these operators. Also, we give some basic results that are connected to this formulation. Note that in the following Sobolev spaces are defined in the complex domain only. This is not specifically indicated in the corresponding symbols. Also, since we use the notation $\| \ \|_1, \| \ \|_2$ for the L^1-norm and L^2-norm, respectively, Sobolev norms will be indicated by the symbol $|\!|\!| \ |\!|\!|$ with the appropriate subscript.

General Assumption 1. *In the following, let $n \in \mathbb{N}^*$ and let Ω be a nonempty open subset of $\mathbb{R}^n$.*

Definition 3.1 (weak derivatives and Sobolev spaces). *We define the following:*

[1] After that, the abstract theory of strongly continuous one-parameter semigroups of operators can be used to associate a rigorous formulation of a well-posed initial value problem to (2.1) [9, 13, 21]. In this, A_p becomes the infinitesimal generator of time evolution. This last step will not be detailed here.

(i) *A scalar product* $\langle\,|\,\rangle_2$ *is on* $L^2_{\mathbb{C}}(\Omega)$ *by*

$$\langle f|g\rangle_2 := \int_\Omega f^* g \, dv^n$$

for all $f, g \in L^2_{\mathbb{C}}(\Omega)$. *Here* * *denotes complex conjugation on* $\mathbb{C}$. *As a consequence,* $\langle\,|\,\rangle_2$ *is antilinear in the first argument and linear in the second. This convention will be used for sesquilinear forms in general.*

(ii) *For every multi-index* $\alpha \in \mathbb{N}^n$ *the densely defined linear operator* ∂^α *is in* $L^2_{\mathbb{C}}(\Omega)$ *by*

$$\partial^\alpha := (-1)^{|\alpha|} . \left(C_0^\infty(\Omega, \mathbb{C}) \to L^2_{\mathbb{C}}(\Omega), f \mapsto \frac{\partial^\alpha f}{\partial x^\alpha} \right)^* ,$$

where $*$ *denotes the adjoint operation and*

$$|\alpha| := \sum_{j=1}^n \alpha_j .$$

(iii) *For every* $k \in \mathbb{N}$ *the complex Sobolev space* $H^k(\Omega)$ *is of order* k *by*

$$H^k(\Omega) := \bigcap_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant k} D(\partial^\alpha) .$$

Equipped with the scalar product

$$\langle\,,\rangle_k \ : (H^k(\Omega))^2 \to \mathbb{C} ,$$

defined by

$$\langle f, g\rangle_k := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leqslant k} \langle \partial^\alpha f | \partial^\alpha g\rangle_2$$

for all $f, g \in H^k(\Omega)$, $H^k(\Omega)$ *becomes a Hilbert space.*

(iv) $H^k_0(\Omega)$ *is the closure of* $C_0^\infty(\Omega, \mathbb{C})$ *in* $(H^k(\Omega), \|\,\|_k)$, *where* $\|\,\|_k$ *denotes the norm that is induced on* $H^k(\Omega)$ *by* $\langle\,,\rangle_k$.

We note the following lemma.

Lemma 3.2 (partial integration).

$$\langle f|\partial^{e_k} g\rangle_2 = -\langle \partial^{e_k} f|g\rangle_2 \tag{3.1}$$

for all $(f, g) \in H^1_0(\Omega) \times H^1(\Omega)$ *and* $k \in \mathbb{N}^*$, *where* e_k *denotes the* k*th canonical unit vector of* $\mathbb{R}^n$.

The next definition defines gradient operators.

Definition 3.3 (gradient operators). *We define the* $(L^2_{\mathbb{C}}(\Omega))^n$*-valued densely defined linear operators in* $L^2_{\mathbb{C}}(\Omega)$

$$\nabla_0 : C_0^\infty(\Omega, \mathbb{C}) \to (L^2_{\mathbb{C}}(\Omega))^n , \quad \nabla_w : H^1(\Omega) \to (L^2_{\mathbb{C}}(\Omega))^n$$

by

$$\nabla_0 f := {}^t\!\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) , \quad \nabla_w g := {}^t(\partial^{e_1} g, \ldots, \partial^{e_n} g)$$

for all $f \in C_0^\infty(\Omega, \mathbb{C})$ *and* $g \in H^1(\Omega)$.

It is easy to see that the following holds.

LEMMA 3.4 (adjoints of gradient operators).

$$(\nabla_0{}^*)^* = \nabla_w\big|_{H_0^1(\Omega)} \ , \quad \left(\nabla_w\big|_{H_0^1(\Omega)}\right)^* = \nabla_0{}^* \ . \tag{3.2}$$

Note that $-\nabla_0{}^*$ can be viewed as a divergence operator.

4. Basic properties of the diffusion operator. This section provides the basis of this paper. It defines the diffusion operator as the operator in $L^2_{\mathbb{C}}(\Omega)$ and gives basic properties.

DEFINITION 4.1. *Let* $p : \Omega \to \mathbb{R}$ *be measurable. We define the linear operator* $A : D(A) \to L^2_{\mathbb{C}}(\Omega)$ *in* $L^2_{\mathbb{C}}(\Omega)$ *by*

$$D(A) := \{u \in H_0^1(\Omega) : p\,\nabla_w u \in D(\nabla_0{}^*)\}$$

and

$$Au := \nabla_0{}^* p\,\nabla_w u$$

for every $u \in D(A)$.

Diffusion operators corresponding to diffusivities from the following large subset $\mathcal{L}$ of $L^\infty(\Omega)$ will turn out to be densely defined, linear, and self-adjoint operators.

DEFINITION 4.2. *We define the subset* $\mathcal{L}$ *of* $L^\infty(\Omega)$ *to consist of those elements* p *for which there are real* C_1, C_2 *satisfying* $C_2 \geqslant C_1 > 0$ *and such that* $C_1 \leqslant p \leqslant C_2$ *a.e. on* Ω. *Note that the last inequality also implies that* $1/p \in \mathcal{L}$ *and, in particular, that* $1/C_2 \leqslant 1/p \leqslant 1/C_1$ *a.e. on* Ω.

The next theorem proves the self-adjointness of diffusion operators corresponding to diffusivities from $\mathcal{L}$. For this, the so-called *form methods* from operator theory are used. For these methods, see [15].

THEOREM 4.3. *Let* $p \in \mathcal{L}$. *Then* A *is a densely defined, linear, and self-adjoint operator in* $L^2_{\mathbb{C}}(\Omega)$.

Proof. For this, we define a positive Hermitian sesquilinear form $s : (H_0^1(\Omega))^2 \to \mathbb{C}$ by

$$s(u, v) := \langle \nabla_w u \,|\, p\,\nabla_w v \rangle_{2,n}$$

for all $u, v \in H_0^1(\Omega)$. Then $\langle\,|\,\rangle_s : (H_0^1(\Omega))^2 \to \mathbb{C}$, defined by

$$\langle u|v \rangle_s := s(u, v) + \langle u|v \rangle_2$$

for every $u, v \in H_0^1(\Omega)$, defines a scalar product on $H_0^1(\Omega)$ with induced norm $\|\ \|_s : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\|u\|_s^2 = \langle \nabla_w u \,|\, p\,\nabla_w u \rangle_{2,n} + \|u\|_2^2$$

for all $u \in H_0^1(\Omega)$. In particular, s is closable. For the proof, let $u_1, u_2, \ldots,$ be a Cauchy sequence in $(H_0^1(\Omega), \|\ \|_s)$ such that

$$\lim_{\nu \to \infty} \|u_\nu\|_2 = 0 \ .$$

We note that

$$\min\{1, C_1\}\, |||u|||_1^2 \leqslant C_1\, \|\nabla_w u\|_{2,n}^2 + \|u\|_2^2 \leqslant \|u\|_s^2$$

$$\leqslant C_2 \, \|\nabla_w u\|_{2,n} + \|u\|_2^2 \leqslant \max\{1, C_2\} \, |||u|||_1^2 \ ,$$

where $C_1, C_2 \in \mathbb{R}$ satisfy $C_2 \geqslant C_1 > 0$ and are such that $C_1 \leqslant p \leqslant C_2$ a.e. on Ω, and hence $\| \ \|_s$ and the restriction of $||| \ |||_1$ to $H_0^1(\Omega)$ are equivalent. Hence it follows that

$$\lim_{\nu \to \infty} \|u_\nu\|_s = 0 \ .$$

Since $(H_0^1(\Omega), \| \ \|_s)$ is, in particular, complete, it follows that s coincides with its closure. As a consequence, there is a unique densely defined, linear, and self-adjoint operator $A : D(A) \to L^2_{\mathbb{C}}(\Omega)$ in $L^2_{\mathbb{C}}(\Omega)$ such that $D(A)$ is a dense subspace of $(H_0^1(\Omega), ||| \ |||_1)$ such that

$$\langle u | Au \rangle_2 = s(u,u) = \langle \nabla_w u \, | \, p \, \nabla_w u \rangle_{2,n}$$

for all $u \in D(A)$. In particular, $D(A)$ consists of all $u \in H_0^1(\Omega)$ for which there is $f \in L^2_{\mathbb{C}}(\Omega)$ such that

$$\langle \, f \, | \ldots \rangle_2 \big|_{H_0^1(\Omega)} = \langle \, p \, \nabla_w u \, | \, \nabla_w \ldots \rangle_{2,n} \big|_{H_0^1(\Omega)} \ .$$

Further, if u and f satisfy these requirements, then

$$Au = f \ .$$

Hence $u \in D(A)$ if and only if

$$p \, \nabla_w u \in D\left(\left(\nabla_w \big|_{H_0^1(\Omega)} \right)^* \right) = D({\nabla_0}^*) \ ,$$

and in this case

$$Au = {\nabla_0}^* \, p \, \nabla_w u \ . \qquad \square$$

For completeness, the next corollary gives the proof that diffusion operators corresponding to bounded Ω and diffusivities from $\mathcal{L}$ have a purely discrete spectrum, i.e., that their spectrum is a discrete subset of the real numbers consisting of eigenvalues of finite multiplicity and that there is a Hilbert basis consisting of eigenvectors. This result is not used in the following.

COROLLARY 4.4. *Let $p \in \mathcal{L}$ and, in addition, Ω be bounded. Then A has a purely discrete spectrum.*

Proof. According to the proof of Theorem 4.3, $\| \ \|_s : H_0^1(\Omega) \to \mathbb{R}$ defines a norm which is equivalent to the restriction of $||| \ |||_1$ to $H_0^1(\Omega)$. Hence the closed unit ball B in $(H_0^1(\Omega), \| \ \|_s)$ is contained in a closed ball of $(H_0^1(\Omega), ||| \ |||_1)$. The closed ball is relatively compact in $L^2_{\mathbb{C}}(\Omega)$. From this, it follows also that B is relatively compact in $L^2_{\mathbb{C}}(\Omega)$. Hence it follows (see, e.g., [24]) that A has a purely discrete spectrum. $\square$

Example 2. The following example illustrates the influence of discontinuities of the diffusivity on the regularity of the elements in $D(A)$. Consider the case that $\Omega = I := (-1, 1)$ and a piecewise constant diffusivity $p : I \to \mathbb{R}$ given by

$$p(x) := \begin{cases} 1 & \text{if } -1 < x < -1/2 \ , \\ M & \text{if } -1/2 \leqslant x \leqslant 1/2 \ , \\ 1 & \text{if } 1/2 < x < 1 \end{cases}$$

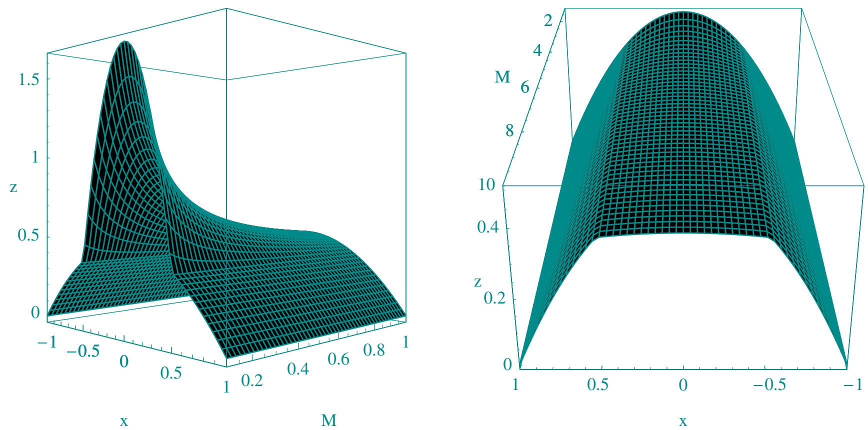

FIG. 1. *Graphs of u from Example 2 as a function of M.*

for $x \in I$, where $M > 0$. Then $Au = f$, where $u : I \to \mathbb{R}$ is defined by

$$u(x) := \begin{cases} (1 - x^2)/2 & \text{if } -1 < x \leqslant -1/2 \ , \\ (1 - 4x^2 + 3M)/(8M) & \text{if } -1/2 < x < 1/2 \ , \\ (1 - x^2)/2 & \text{if } 1/2 \leqslant x < 1 \ , \end{cases}$$

and f is the constant function on I of value 1. The solutions u are depicted for varying M values in Figure 1. We note that u' has no extension to a continuous function on I if $M \neq 1$. In general, discontinuities in the diffusivity cause low regularity of elements in $D(A)$. Also, see the concluding remarks.

There is a unique solution u_f to

$$Au_f = f$$

for every $f \in L^2_{\mathbb{C}}(\Omega)$ if and only if A is bijective or, equivalently, if and only if 0 is not part of the spectrum of A. In general, A is not bijective. For instance, the operator A that is associated to $\Omega = \mathbb{R}^n$ and the diffusivity $p(x) = 1$ for every $x \in \mathbb{R}^n$ is not surjective. Below, we place a restriction on Ω that leads to bijective diffusion operators.

GENERAL ASSUMPTION 3. *In the following, we assume that Ω is in addition such that the following Poincare inequality is valid:*

$$\| \partial^{e_j} f \|_2 \geqslant c \, \| f \|_2 \tag{4.1}$$

for some $j \in \{1, \dots, n\}$ and every $f \in H^1_0(\Omega)$, where $c > 0$. In the remainder, such c is considered chosen.

Remark 4. It is known that such Ω are not necessarily bounded. For instance, every nontrivial open set, for which there is $\mathbf{n} \in \mathbb{R}^n \backslash \{0\}$ along with real numbers a, b such that

$$a < x \cdot \mathbf{n} < b$$

for all $x \in \Omega$, is of this type.

In particular, the following theorem proves that diffusion operators corresponding to diffusivities from $\mathcal{L}$ are bijective.

THEOREM 4.5. *Let $p \in \mathcal{L}$. The spectrum $\sigma(A)$ of A satisfies*

$$\sigma(A) \subset [\, c^2 C, \infty) \ , \tag{4.2}$$

where $C > 0$ is such that $p \geqslant C$ a.e. on Ω.

Proof. For $u \in D(A)$, it follows that

$$\langle u | Au \rangle_2 = \langle \nabla_w u \,|\, p\, \nabla_w u \rangle_{2,n} \geqslant C \, \| \nabla_w u \|_{2,n}^2 \geqslant c^2 \, C \, \| u \|_2^2 \ ,$$

where $C > 0$ is such that $p \geqslant C$ a.e. on Ω. Hence it follows the validity of (4.2). □

5. Properties of a first-order operator connected to the diffusion operator. As indicated by Example 2, for nonsmooth diffusivities p, the condition that $p \nabla_w u \in H^1(\Omega)$ in the definition of the domain of A leads to a strong dependence of that domain on the diffusivity. This fact poses an obstacle to the study of the map, associating to every diffusivity $p \in \mathcal{L}$ the corresponding operator A^{-1}, by the notion of strong resolvent convergence; see [22, section VIII.7], [15, section VIII.1]. By use of the following vector partial differential operator $\hat{A}$ of the first order, this problem can be circumvented. The operator $\hat{A}$ is defined in terms of a function ϖ with a domain being independent of ϖ. The connection of the resolvents of A and $\hat{A}$ is given in Theorem 5.5. In this, the corresponding function ϖ and the diffusivity p are related by $\varpi = 1/p$. As a consequence, with the help of $\hat{A}$, limits of A^{-1} with diffusivities approaching infinity on subdomains of Ω can be studied more easily.

DEFINITION 5.1. *Let $\varpi \in L^\infty(\Omega)$. We define the densely defined, linear operator $\hat{A} : H_0^1(\Omega) \times D({\nabla_0}^*) \to L_{\mathbb{C}}^2(\Omega) \times (L_{\mathbb{C}}^2(\Omega))^n$ in $L_{\mathbb{C}}^2(\Omega) \times (L_{\mathbb{C}}^2(\Omega))^n$ by*

$$\hat{A}(u, q) := \big({\nabla_0}^* q \, , \, \nabla_w u - \varpi \, q \big)$$

for every $(u, q) \in H_0^1(\Omega) \times D({\nabla_0}^*)$.

THEOREM 5.2. *The operator $\hat{A}$ is self-adjoint.*

Proof. The statement is an immediate consequence of Lemma 3.4. □

The following gives a characterization of the kernel of $\hat{A}$. In particular, the result implies that $\hat{A}$ is bijective for $\varpi \in \mathcal{L}$.

THEOREM 5.3. *Let $\varpi \in L^\infty(\Omega)$ be a.e. positive. Then*

$$\ker \hat{A} = \{0\} \times (\, \ker {\nabla_0}^* \cap \ker T_\varpi^n \,) \ ,$$

where $T_\varpi \in L(L_{\mathbb{C}}^2(\Omega), L^2(\Omega))$ denotes the maximal multiplication operator in $L_{\mathbb{C}}^2(\Omega)$ that is associated to ϖ.

Proof. "$\subset$": The proof is obvious. "$\supset$": Let $(u, q) \in \ker \hat{A}$. Then

$$\begin{aligned} {\nabla_0}^* q &= 0 \, , \\ \nabla_w u - \varpi \, q &= 0, \end{aligned} \tag{5.1}$$

and hence

$$\begin{aligned} 0 &= \langle \, q \,|\, \nabla_w u - \varpi \, q \, \rangle_{2,n} = \langle \, q \,|\, \nabla_w u \rangle_{2,n} - \langle \, q \,|\, \varpi \, q \, \rangle_{2,n} \\ &= \langle \, {\nabla_0}^* q \,|\, u \, \rangle_2 - \| \, \varpi^{1/2} q \, \|_{2,n}^2 = - \| \, \varpi^{1/2} q \, \|_{2,n}^2 \ . \end{aligned}$$

The last equation implies that

$$q \in \ker T_{\varpi^{1/2}}^n \ ,$$

where $T_{\varpi^{1/2}} \in L(L^2_{\mathbb{C}}(\Omega), L^2_{\mathbb{C}}(\Omega))$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\Omega)$ that is associated to $\varpi^{1/2}$, and hence also that

$$q \in \ker T^n_{\varpi} \ .$$

Further, by (5.1), it follows that

$$\nabla_w u = 0 \ .$$

The last equation implies that

$$\nabla_0{}^* \nabla_w u = 0 \ ,$$

and hence by Theorem 4.5 that $u = 0$. □

The following example shows that the kernel of $\hat{A}$ is nontrivial if ϖ vanishes on some open subset of Ω. The vanishing of ϖ on nonempty subsets of Ω corresponds to the asymptotic cases mentioned in the introduction.

Example 5. In the following, we give $q \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n) \cap \ker \nabla_0{}^*$ for $n \geqslant 2$. For this, let h be an element of $C_0^\infty(\mathbb{R})$ with support contained in $[-1, 1]$. In addition, let α be a nonzero antisymmetric $n \times n$-matrix. We define $q \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ by

$$q(x) := \frac{h(|x|^2)}{2} \sum_{i,j=1}^{n} \alpha_{ij} x_j e_i$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\sum_{i=1}^{n} \frac{\partial q_i}{\partial x_i}(x) = h'(|x|^2) \sum_{i,j=1}^{n} \alpha_{ij} x_i x_j = 0$$

for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and hence $q \in \ker \nabla_0{}^*$.

The following lemma prepares the subsequent theorem which estimates the size of the gap around 0 in the spectrum of A and gives a representation of the resolvent of $\hat{A}$ in terms of the resolvent of A, i.e., (5.2). The main parts of Lemma 5.4 can be viewed as elliptic regularity estimates. Roughly speaking, in the theory of unbounded operators on Banach spaces, the closedness of operators replaces continuity as a tool. Additional control of the domains of involved operators often allows the application of the closed graph theorem which results in such estimates. On the other hand, usually, such an approach does not give an estimate on the size of the involved constants. The subsequent Theorem 5.5 works around this. The proofs of (i), (ii), and (iv) are rather straightforward applications of the closed graph theorem in the form of Theorem 3.1.9 in [9]. For this reason, the corresponding proofs are omitted.

Lemma 5.4. *Let $p \in \mathcal{L}$, let $\sigma(A)$ be the spectrum of A, let $\lambda < \min\{\sigma(A)\}$, and let $A_\lambda := A - \lambda$.*[2] *Then*

(i) $\overline{\nabla_w A_\lambda^{-1}} \in L(L^2_{\mathbb{C}}(\Omega), (L^2_{\mathbb{C}}(\Omega))^n)$,

(ii) $\overline{A_\lambda^{-1} \nabla_0{}^*} = (\, \nabla_w A_\lambda^{-1}\,)^*$,

(iii) *$D(A_\lambda^{1/2}) = H^1_0(\Omega)$ and $A_\lambda^{1/2} : H^1_0(\Omega) \to L^2_{\mathbb{C}}(\Omega)$ is continuous,*

(iv) *$\overline{\nabla_w A_\lambda^{-1} \nabla_0{}^*}$ is a positive self-adjoint element of $L((L^2_{\mathbb{C}}(\Omega))^n, (L^2_{\mathbb{C}}(\Omega))^n)$.*

[2]For any operator A with domain and range in some vector space X over a field $\mathbb{K}$ and $\lambda \in \mathbb{K}$, we use the short notation $A - \lambda$ for $A - \lambda\, \mathrm{id}_X := (D(A) \to X, \xi \mapsto A\xi - \lambda\xi)$.

Proof. Lemma 5.4(iii): In a first step, we prove the statement for the case $\lambda = 0$. For this, we note that, as a consequence of Theorem 4.5, $0 < \min\{\sigma(A)\}$. Further, we note that $D(A)$ is a core $A^{1/2}$. For instance, this follows by Theorem 3.1.9 in [9]. Hence $D(A)$ is dense in the Banach space $(D(A^{1/2}), \| \, \|_{A^{1/2}})$, where

$$\|f\|_{A^{1/2}} := \left[\, \|f\|_2^2 + \|A^{1/2}f\|_2^2 \, \right]^{1/2}$$

for every $f \in D(A^{1/2})$. Further, it follows for $f \in D(A)$ that

$$\|A^{1/2}f\|_2^2 = \langle \, f \mid Af \, \rangle_2 = \langle \, \nabla_w f \mid p \, \nabla_w f \, \rangle_{2,n} = s(f,f) \ ,$$

where the real numbers C_1, C_2 and the sesquilinear form s are as in the proof of Theorem 4.3, and hence that

$$\min\{1, C_1\} \, |||f|||_1^2 \leqslant \|f\|_{A^{1/2}}^2 \leqslant \max\{1, C_2\} \, |||f|||_1^2 \ .$$

As a consequence, the restrictions of $\| \, \|_{A^{1/2}}$ and $||| \, |||_1$ to $D(A)$ are equivalent. Since $D(A)$ is dense in $(D(A^{1/2}), \| \, \|_{A^{1/2}})$, it follows for $f \in D(A^{1/2})$ that the existence of a sequence $f_1, f_2, \dots$ in $D(A)$ is such that

$$\lim_{\nu \to \infty} \|f_\nu - f\|_{A^{1/2}} = 0 \ .$$

Since the inclusion of $(D(A^{1/2}), \| \, \|_{A^{1/2}})$ into $L^2_{\mathbb{C}}(\Omega)$ is continuous, this implies also that

$$\lim_{\nu \to \infty} \|f_\nu - f\|_2 = 0 \ .$$

Since the restrictions of $\| \, \|_{A^{1/2}}$ and $||| \, |||_1$ to $D(A)$ are equivalent, it follows that $f_1, f_2, \dots$ is a Cauchy sequence in $H^1_0(\Omega)$, and hence convergent to some $\bar{f} \in H^1_0(\Omega)$. Since the embedding of $(H^1(\Omega), ||| \, |||_1)$ into $L^2_{\mathbb{C}}(\Omega)$ is continuous, it also follows that

$$\lim_{\nu \to \infty} \|f_\nu - \bar{f}\,\|_2 = 0,$$

and hence that $f = \bar{f} \in H^1_0(\Omega)$. Further, it follows that

$$\min\{1, C_1\} \, |||f|||_1^2 \leqslant \|f\|_{A^{1/2}}^2 \leqslant \max\{1, C_2\} \, |||f|||_1^2,$$

and hence that $\| \, \|_{A^{1/2}}$ and the restriction of $||| \, |||_1$ to $D(A^{1/2})$ are equivalent. Since, according to the proof of Theorem 4.3, $D(A)$ is a dense subspace of $(H^1_0(\Omega), ||| \, |||_1)$, we conclude that $D(A^{1/2}) = H^1_0(\Omega)$, and that $A^{1/2} : H^1_0(\Omega) \to L^2_{\mathbb{C}}(\Omega)$ is continuous. From this, we conclude that the statement of Lemma 5.4(ii) is as follows. For this, let $\Lambda \in \mathbb{R} \backslash \sigma(A)$ be such that $\Lambda > \max\{0, \lambda\}$. Since $\mathbb{R} \backslash \sigma(A)$ is open, such Λ exists. We note that $D(A)$ is a core also for $A_\lambda^{1/2}$ and $A_\Lambda^{1/2}$. For instance, this follows by Theorem 3.1.9 in [9]. Hence $D(A)$ is dense in the Banach spaces $(D(A_\lambda^{1/2}), \| \, \|_{A_\lambda^{1/2}})$, $(D(A_\Lambda^{1/2}), \| \, \|_{A_\Lambda^{1/2}})$, where

$$\|f\|_{A_\lambda^{1/2}} := \left[\, \|f\|_2^2 + \|A_\lambda^{1/2}f\|_2^2 \, \right]^{1/2} \ , \quad \|g\|_{A_\Lambda^{1/2}} := \left[\, \|g\|_2^2 + \|A_\Lambda^{1/2}g\|_2^2 \, \right]^{1/2}$$

for all $f \in D(A_\lambda^{1/2})$ and $g \in D(A_\Lambda^{1/2})$. Further, it follows for every $f \in D(A)$ that

$$\|f\|_{A_\lambda^{1/2}}^2 = \|A_\lambda^{1/2}f\|_2^2 + \|f\|_2^2 = \langle f | A_\lambda f \rangle_2 + \|f\|_2^2$$

$$= \langle f|A_\Lambda f\rangle_2 + \|f\|_2^2 + (\Lambda-\lambda)\|f\|_2^2 = \|f\|^2_{A_\Lambda^{1/2}} + (\Lambda-\lambda)\|f\|_2^2\ ,$$

and hence that

$$\|f\|^2_{A_\lambda^{1/2}} \geqslant \|f\|^2_{A_\Lambda^{1/2}}$$

as well as that

$$\|f\|^2_{A_\lambda^{1/2}} \leqslant [1+(\Lambda-\lambda)]\,\|f\|^2_{A_\Lambda^{1/2}}\ .$$

Since $D(A)$ is dense in the Banach spaces $(D(A_\lambda^{1/2}), \|\ \|_{A_\lambda^{1/2}})$ and $(D(A_\Lambda^{1/2}), \|\ \|_{A_\Lambda^{1/2}})$, it follows that

$$D(A_\lambda^{1/2}) = D(A_\Lambda^{1/2})$$

as well as the equivalence of the norms $\|\ \|_{A_\lambda^{1/2}}$ and $\|\ \|_{A_\Lambda^{1/2}}$. In particular, this implies that

$$D(A_\lambda^{1/2}) = D(A^{1/2})$$

and the equivalence of the norms $\|\ \|_{A_\lambda^{1/2}}$ and $\|\ \|_{A^{1/2}}$. By this, statement (ii) follows from the corresponding statement of (ii) for the special case that $\lambda = 0$. ∎

With the help of the previous lemma, we can now estimate the size of the gap around 0 in the spectrum of A and give a representation of the resolvent of $\hat{A}$ in terms of the resolvent of A, i.e., (5.2). The proof proceeds by direct calculation and is omitted.

Theorem 5.5. *Let $\varpi \in \mathcal{L}$, $C_1, C_2 \in \mathbb{R}$ satisfy $C_2 \geqslant C_1 > 0$ and be such that $C_1 \leqslant \varpi \leqslant C_2$ a.e. on Ω. Then the interval*

$$J := (\,-C_1\,,\,c^2/(c+C_2)\,)$$

is contained in the resolvent set of $\hat{A}$. In particular for $\lambda \in J$, $(\hat{A}-\lambda)^{-1}$ is given by

$$\begin{aligned}&(\hat{A}-\lambda)^{-1}(f,g)\\ &= \left((A_{p_\lambda}-\lambda)^{-1}\,,\,p_\lambda\nabla_w(A_{p_\lambda}-\lambda)^{-1}\right)(f)\\ &+\left(\overline{(A_{p_\lambda}-\lambda)^{-1}\nabla_0{}^*}\,p_\lambda\,,\,-p_\lambda + p_\lambda\overline{\nabla_w(A_{p_\lambda}-\lambda)^{-1}\nabla_0{}^*}\,p_\lambda\right)(g)\end{aligned} \tag{5.2}$$

for all $(f,g) \in L^2_{\mathbb{C}}(\Omega)\times(L^2_{\mathbb{C}}(\Omega))^n$, where $p_\lambda := 1/(\varpi+\lambda)$, and A_{p_λ} is the operator corresponding to p_λ according to Definition 4.1.

With the help of the previous theorem, the next result follows by the application of a well-known criterion for the strong resolvent convergence of sequences of self-adjoint operators.

Theorem 5.6. *Let $\varpi_1, \varpi_2, \dots$ be a uniformly bounded sequence in $\mathcal{L}$ for which there is $\varepsilon > 0$ such that $\varpi_\nu \geqslant \varepsilon$ for all $\nu \in \mathbb{N}^*$ and which converges a.e. pointwise on Ω to $\varpi_\infty \in \mathcal{L}$. In addition, let $\hat{A}_1, \hat{A}_2, \dots$ be the associated sequence of self-adjoint linear operators and $\hat{A}_\infty$ be the self-adjoint linear operator associated to ϖ_∞. Then*

$$s-\lim_{\nu\to\infty}\hat{A}_\nu^{-1} = \hat{A}_\infty^{-1}\ .$$

Proof. By application of Lebesgue's dominated convergence theorem, it follows that

$$\lim_{\nu\to\infty}\|\hat{A}_\nu(u,q) - \hat{A}_\infty(u,q)\| = 0$$

for all $(u, q) \in H_0^1(\Omega) \times D(\nabla_0^*)$, where $\| \ \|$ denotes the norm on $L^2_{\mathbb{C}}(\Omega) \times (L^2_{\mathbb{C}}(\Omega))^n$. From this, the statement follows from a well-known criterion for strong resolvent convergence of a sequence of self-adjoint linear operators; e.g., see part (i) of Theorem 9.16 in [25]. □

COROLLARY 5.7. *Let $p_1, p_2, \ldots$ be a uniformly bounded sequence in $\mathcal{L}$ for which there is $\varepsilon > 0$ such that $p_\nu \geqslant \varepsilon$ for all $\nu \in \mathbb{N}^*$ and which converges a.e. pointwise on Ω to $p_\infty \in \mathcal{L}$. In addition, let $A_1, A_2, \ldots$ be the associated sequence of self-adjoint linear operators, and let A_∞ be the self-adjoint linear operator associated to p_∞. Finally, let $f \in L^2_{\mathbb{C}}(\Omega)$. Then*

$$\lim_{\nu\to\infty} \|A_\nu^{-1} f - A_\infty^{-1} f\|_2 = \lim_{\nu\to\infty} \|p_\nu \nabla_w A_\nu^{-1} f - p_\infty \nabla_w A_\infty^{-1} f\|_{2,n} = 0 \ . \tag{5.3}$$

Proof. By Theorem 5.6, it follows that

$$\lim_{\nu\to\infty} \hat{A}_\nu^{-1}(f, 0) = \hat{A}_\infty^{-1}(f, 0) \ ,$$

where $\hat{A}_1, \hat{A}_2, \ldots$ is the associated sequence of self-adjoint linear operators to $1/p_1, 1/p_2$, $\ldots$ and $\hat{A}_\infty$ is the self-adjoint linear operator associated to $1/p_\infty$. Hence (5.3) follows by Theorem 5.5. □

An alternative proof of Corollary 5.7 will be given below.

DEFINITION 5.8 (weak solutions). *Let X be a nontrivial complex Hilbert space, and let $A : D(A) \to X$ be a densely defined, linear, self-adjoint, and strictly positive operator in X. For $\eta \in X$, we call $\xi \in D(A^{1/2})$ a weak solution of*

$$A\xi = \eta \tag{5.4}$$

if

$$\langle A^{1/2}\xi | A^{1/2}\xi' \rangle = \langle \eta | \xi' \rangle$$

for every $\xi' \in D(A^{1/2})$.

Remark 6. We note that the *strong* solution of (5.4), $\xi := A^{-1}\eta$, is also a weak solution of that equation. In addition, by the bijectivity of $A^{1/2}$, it follows the uniqueness of a weak solution. Hence $\xi \in D(A^{1/2})$ is a weak solution of (5.4) if and only if it is a *strong* solution of (5.4), i.e., if and only if $\xi \in D(A)$ and $A\xi = \eta$.

THEOREM 5.9 (weak formulation for the inversion of A). *Let $p \in \mathcal{L}$, let A be the associated self-adjoint linear operator, and let $f \in L^2_{\mathbb{C}}(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H_0^1(\Omega)$ and*

$$\langle \nabla_w u \,|\, p \nabla_w g \rangle_{2,n} = \langle f | g \rangle_2$$

for every $g \in H_0^1(\Omega)$.

Proof. According to Definition 5.8 and Lemma 5.4(iii), $u \in D(A)$ and $Au = f$ if and only if $u \in H_0^1(\Omega)$ and

$$\langle A^{1/2} u | A^{1/2} g \rangle = \langle f | g \rangle_2$$

for every $g \in H_0^1(\Omega)$. Further, according to Lemma 5.4(iii) and the corresponding proof, $A^{1/2} : H_0^1(\Omega) \to L^2_{\mathbb{C}}(\Omega)$ is continuous, $\| \ \|_{A^{1/2}}$ and $||| \ |||_1$ are equivalent, and

$$\|A^{1/2} g\|_2^2 = \langle \nabla_w g \,|\, p \, \nabla_w g \rangle_{2,n}$$

for every $g \in D(A)$. Hence, since $D(A)$ is a core for $A^{1/2}$, we conclude that

$$\|A^{1/2}g\|_2^2 = \langle \nabla_w g \,|\, p\, \nabla_w g \rangle_{2,n}$$

for every $g \in H_0^1(\Omega)$ and by polarization that

$$\langle A^{1/2}g|A^{1/2}h\rangle = \langle \nabla_w g \,|\, p\, \nabla_w h \rangle_{2,n}$$

for all $g, h \in H_0^1(\Omega)$. □

Now, we give the alternative a second proof of Corollary 5.7.

Proof. For this, let $\nu \in \mathbb{N}^*$, let $u_\nu := A_\nu^{-1} f$, and let $u_\infty := A_\infty^{-1} f$. Then,

$$\langle \nabla_w u_\nu | p_\nu \nabla_w g\rangle_{2,n} = \langle f|g\rangle_2 = \langle \nabla_w u_\infty | p_\infty \nabla_w g\rangle_{2,n}$$

for all $g \in H_0^1(\Omega)$. Hence

$$\begin{aligned}&\varepsilon\|\nabla_w(u_\nu - u_\infty)\|_{2,n}^2 \leqslant \langle \nabla_w(u_\nu - u_\infty)|p_\nu \nabla_w(u_\nu - u_\infty)\rangle_{2,n}\\ &= \langle \nabla_w u_\infty|(p_\infty - p_\nu)\nabla_w(u_\nu - u_\infty)\rangle_{2,n}\end{aligned}$$

and

$$\|\nabla_w(u_\nu - u_\infty)\|_{2,n} \leqslant \varepsilon^{-1}\|(p_\infty - p_\nu)\nabla_w u_\infty\|_{2,n}.$$

Since

$$\begin{aligned}&\|u_\nu - u_\infty\|_2 \leqslant c^{-1}\|\nabla_w(u_\nu - u_\infty)\|_{2,n}\ ,\\ &\|p_\nu \nabla_w u_\nu - p_\infty \nabla_w u_\infty\|_{2,n} \leqslant C\|\nabla_w(u_\nu - u_\infty)\|_{2,n} + \|(p_\nu - p_\infty)\nabla_w u_\infty\|_{2,n}\end{aligned}$$

for some $C \geqslant 0$, (5.3) follows by application of the dominated convergence theorem. □

6. The one-dimensional case. In the specialcases that Ω is given by a nonempty bounded open interval I of $\mathbb{R}$ and $\varpi \in L^\infty(\Omega)$ is a.e. positive, but not a.e. vanishing on I, $\hat{A}^{-1}$ turns out to be bijective. As a consequence, by use of the methods of section 5, an analogue of Theorem 5.6 could be derived, where $\mathcal{L}$ is replaced by the subset of $L_\infty(I)\backslash\{0\}$ containing only a.e. positive elements. In addition, an analogue of Corollary 5.7 would give rise to a result that also includes sequences of diffusivities that approach infinity on subsets of I. These sequences of diffisuvities are most interesting for the purpose of preconditioning. On the other hand, a generalization of the alternative proof of Corollary 5.7, based on weak solutions, appears impossible.

In the following, we don't pursue such an approach because $\hat{A}^{-1}$ can be explicitly calculated for these cases. As a result, the use of direct methods leads to stronger results. The fact that $\hat{A}^{-1}$ can be explicitly calculated is somewhat surprising since in this case the corresponding A is a Sturm–Liouville operator and the standard method of calculating its inverse (e.g., see Theorem 8.26 in [25]) appears not directly applicable for general $p \in \mathcal{L}$. Still, by direct calculation of $\hat{A}^{-1}$, an explicit expression for A^{-1} can be given with the help of (5.2).

THEOREM 6.1. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\varpi \in L^\infty(\Omega)\backslash\{0\}$ be a.e. positive. Then $\hat{A}$ is bijective and has a purely discrete spectrum. In particular, $\hat{A}^{-1}$ is given by*

$$(u, q) := \hat{A}^{-1}(f, g)$$

for every $(f,g) \in (L^2_{\mathbb{C}}(I))^2$*, where*

$$u(x) = \int_a^x g(y)\,dy + \int_a^x \left[\int_y^x \varpi(u)\,du\right] f(y)\,dy$$
$$+\,\|\varpi\|_1^{-1} \int_a^x \varpi(y)\,dy \left\{\int_a^b \left[\int_y^b \varpi(x)\,dx\right] f(y)\,dy - \int_a^b g(y)\,dy\right\}$$
$$q(x) = -\int_a^x f(y)\,dy + \|\varpi\|_1^{-1} \left\{\int_a^b \left[\int_y^b \varpi(x)\,dx\right] f(y)\,dy - \int_a^b g(y)\,dy\right\}$$

for every $x \in I$*. Also,* $\hat{A}^{-1}$ *satisfies*

$$\|\hat{A}^{-1}\| \leqslant 2\,(b-a)\,\|\varpi\|_1^{-1}(\,1 + \|\varpi\|_1)^2 \ .$$

Proof. For this, we define the derivative operator

$$D_I : C_0^\infty(I,\,\mathbb{C}) \to L^2_{\mathbb{C}}(I)$$

by $D_I f := f'$ for every $f \in C_0^\infty(I,\,\mathbb{C})$. In a first step, for $f \in L^2_{\mathbb{C}}(I)$, $h \in C(\bar{I},\mathbb{C})$ is defined by

$$h(x) := \int_a^x f(y)\,dy$$

for every $x \in I$, and it follows that

$$h \in H^1(I) \text{ and } D_I^* h = -f \ .$$

With the help of this auxiliary result, we proceed with the proof of the lemma. For this, we define for every $(f,g) \in (L^2_{\mathbb{C}}(I))^2$ a corresponding $B(f,g) = (u,q)$ by

$$q(x) := q_0(x) + c \ , \quad u(x) := \int_a^x [\,g(y) + \varpi(y)(q_0(y) + c)]\,dy \ ,$$

where

$$q_0(x) := -\int_a^x f(y)\,dy \ , \quad c := -\|\varpi\|_1^{-1} \int_a^b [\,g(y) + \varpi(y) q_0(y)\,]\,dy$$

for every $x \in I$. With the help of the auxiliary result above, it follows that $(u,q) \in (H^1(I) \cap C(\bar{I},\mathbb{C})) \times D(D_I^*)$ and that

$$D_I^* q = f \ , \quad -D_I^* u - \varpi q = g + \varpi q - \varpi q = g \ .$$

In addition,

$$u_b = \int_a^b [\,g(y) + \varpi(y)(q_0(y) + c)\,]\,dy$$
$$= \int_a^b (\,g(y) + \varpi(y) q_0(y))\,dy + c \int_a^b \varpi(y)\,dy = 0 \ .$$

As a consequence,

$$u_a = u_b = 0 \ .$$

From the last equation, it also follows that $u \in H^1_0(I)$ and further that $(u,q) \in D(\hat{A})$ and

$$\hat{A}B(f,g) = \hat{A}(u,q) = (f,g)\ .$$

Further, we conclude by Fubini's theorem that

$$\begin{aligned} c &= -\|\varpi\|_1^{-1}\int_a^b g(y)\,dy + \|\varpi\|_1^{-1}\int_a^b \varpi(x)\left[\int_a^x f(y)\,dy\right]dx \\ &= \|\varpi\|_1^{-1}\left\{\int_a^b\left[\int_y^b \varpi(x)\,dx\right]f(y)\,dy - \int_a^b g(y)\,dy\right\}\ . \end{aligned}$$

This implies that

$$q(x) = -\int_a^x f(y)\,dy + \|\varpi\|_1^{-1}\left\{\int_a^b\left[\int_y^b \varpi(x)\,dx\right]f(y)\,dy - \int_a^b g(y)\,dy\right\}$$

for every $x \in I$. Further, again by Fubini's theorem, it follows that

$$\begin{aligned} u(x) &= \int_a^x g(y)\,dy + \int_a^x \varpi(y)\,q_0(y)\,dy + c\int_a^x \varpi(y)\,dy \\ &= \int_a^x g(y)\,dy + \int_a^x \varpi(u)\left[\int_a^u f(y)\,dy\right]du + c\int_a^x \varpi(y)\,dy \\ &= \int_a^x g(y)\,dy + \int_a^x\left[\int_y^x \varpi(u)\,du\right]f(y)\,dy \\ &\quad + \|\varpi\|_1^{-1}\int_a^x \varpi(y)\,dy\ \left\{\int_a^b\left[\int_y^b \varpi(x)\,dx\right]f(y)\,dy - \int_a^b g(y)\,dy\right\} \end{aligned}$$

for every $x \in I$. In addition, by Holder's inequality, we conclude that

$$\begin{aligned} |u(x)| &\leqslant 2\,(b-a)^{1/2}\left[\,\|\varpi\|_1\,\|f\|_2 + \|g\|_2\,\right] \leqslant 2\,(b-a)^{1/2}\,(\,1+\|\varpi\|_1)\,\|(f,g)\| \\ |q(x)| &\leqslant (b-a)^{1/2}\left[\,2\,\|f\|_2 + \|\varpi\|_1^{-1}\|g\|_2\,\right] \\ &\leqslant 2\,(b-a)^{1/2}\,\|\varpi\|_1^{-1}\,(\,1+\|\varpi\|_1)\,\|(f,g)\|_2 \end{aligned}$$

for every $x \in I$. The last inequality implies

$$\|u\|_2 \leqslant 2\,(b-a)\,(\,1+\|\varpi\|_1)\,\|(f,g)\|\ ,\quad \|q\|_2 \leqslant 2\,(b-a)\,\|\varpi\|_1^{-1}\,(\,1+\|\varpi\|_1)\,\|(f,g)\|_2$$

and

$$\|(u,q)\| \leqslant 2\,(b-a)\,\|\varpi\|_1^{-1}(\,1+\|\varpi\|_1)^2\,\|(f,g)\|_2\ .$$

As a consequence, by $((L^2_{\mathbb{C}}(I))^2 \to (L^2_{\mathbb{C}}(I))^2, (f,g) \to B(f,g))$, there is defined a compact bounded linear operator B. Since

$$\hat{A}B(f,g) = (f,g)$$

for every $(f,g) \in (L^2_{\mathbb{C}}(I))^2$, the bijectivity of $\hat{A}$ follows as well as that of $\hat{A}^{-1} = B$. Finally, since $\hat{A}^{-1}$ is compact, $\hat{A}$ has a purely discrete spectrum. □

THEOREM 6.2. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\varpi_1, \varpi_2 \in L^{\infty}(\Omega)\backslash\{0\}$ be a.e. positive and $\hat{A}_1, \hat{A}_2$ be the corresponding operators. Then*

$$\|\hat{A}_1^{-1} - \hat{A}_2^{-1}\| \leqslant \frac{2(b-a)}{\|\varpi_1\|_1} \left(2 + \|\varpi_1\|_1 + \|\varpi_2\|_1 + \frac{1}{\|\varpi_2\|_1}\right) \|\, \varpi_2 - \varpi_1\|_1 \ .$$

Proof. The proof proceeds by direct calculation. □

Note that from the last calculation, it is easy to obtain an estimate that is symmetric in ϖ_1, ϖ_2.

COROLLARY 6.3. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\Omega := I := (a, b)$. Further, let $\varpi_{\infty} \in L^{\infty}(\Omega)\backslash\{0\}$ be a.e. positive. Let $\varpi_1, \varpi_2, \dots$ be a sequence of a.e. positive elements of $L^{\infty}(\Omega)\backslash\{0\}$ such that*

$$\lim_{\nu \to \infty} \|\varpi_{\nu} - \varpi_{\infty}\|_1 = 0 \ .$$

In addition, let $\hat{A}_1, \hat{A}_2, \dots$ be the associated sequence of self-adjoint linear operators, and let $\hat{A}_{\infty}$ be the self-adjoint linear operator associated to ϖ_{∞}. Then

$$\lim_{\nu \to \infty} \|\hat{A}_{\nu}^{-1} - \hat{A}_{\infty}^{-1}\| = 0 \ .$$

Proof. The statement is a simple consequence of Theorem 6.2. □

COROLLARY 6.4. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\Omega := I := (a, b)$ and $f \in L^2_{\mathbb{C}}(I)$. Further, let $\varpi_{\infty} \in L^{\infty}(\Omega)\backslash\{0\}$ be a.e. positive and $\varpi_1, \varpi_2, \dots$ be a sequence in $\mathcal{L}$ such that*

$$\lim_{\nu \to \infty} \|\varpi_{\nu} - \varpi_{\infty}\|_1 = 0 \ .$$

In addition, let $A_1, A_2, \dots$ be the sequence of self-adjoint linear operators that is associated to $\varpi_1, \varpi_2, \dots$ and $p_{\nu} := 1/\varpi_{\nu}$ for $\nu \in \mathbb{N}^*$. Then $A_1^{-1}, A_2^{-1}, \dots$ and $-p_1 D_I^* A_1^{-1}$, $-p_2 D_I^* A_2^{-1}, \dots$ are convergent in $L(L^2_{\mathbb{C}}(I), L^2_{\mathbb{C}}(I))$ to $B, C \in L(L^2_{\mathbb{C}}(I), L^2_{\mathbb{C}}(I))$, respectively. In particular, B and C are given by

$$\begin{aligned}
(Bf)(x) &= \int_a^x \left[\int_y^x \varpi_{\infty}(u)\, du \right] f(y)\, dy \\
&\qquad + \|\varpi_{\infty}\|_1^{-1} \int_a^x \varpi_{\infty}(y)\, dy \ \int_a^b \left[\int_y^b \varpi_{\infty}(x)\, dx \right] f(y)\, dy \ , \\
(Cf)(x) &= -\int_a^x f(y)\, dy + \|\varpi_{\infty}\|_1^{-1} \int_a^b \left[\int_y^b \varpi_{\infty}(x)\, dx \right] f(y)\, dy
\end{aligned}$$

for all $x \in I$ and every $f \in L^2_{\mathbb{C}}(I)$.

Proof. The statement is a simple consequence of Theorems 5.5 and 6.1. □

7. Concluding remarks.

7.1. Connections to preconditioning. By assuming a weak notion of convergence in $\mathcal{L}$, in Corollary 5.7, we showed that the maps $\mathcal{S}$ and $\mathcal{T}$ defined by

$$\begin{aligned}
\mathcal{S}(\varpi) &:= A_{1/\varpi}^{-1} \, , \\
\mathcal{T}(\varpi) &:= -\,(1/\varpi)\ \nabla_w A_{1/\varpi}^{-1}
\end{aligned}$$

for every $\varpi \in \mathcal{L}$ are strongly sequentially continuous. For the case $n = 1$ and bounded open intervals of $\mathbb{R}$, we were able to show stronger results that also include the asymptotic cases. We showed that $\mathcal{S}$ and $\mathcal{T}$ have unique extensions to sequentially continuous maps $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ in the operator norm on the set of a.e. positive elements of $L^\infty(\Omega)\backslash\{0\}$. In Theorem 6.2, we also gave an estimate of the convergence behavior of the maps. In particular,

$$\hat{\mathcal{S}}(\varpi) = \hat{\mathcal{S}}(\varpi_\infty) + \mathcal{O}(\|\varpi - \varpi_\infty\|_1)\,. \tag{7.1}$$

It is currently unknown if the result (7.1) holds for $n > 1$. This is an avenue for future research. However, for $n = 1$, in Theorem 6.1, we explicitly calculated $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$. The knowledge of $\hat{\mathcal{S}}$ and $\hat{\mathcal{T}}$ for asymptotic ϖ is essential for the purpose of preconditioning. Since $\hat{\mathcal{S}}$ maintains continuity on $\partial\mathcal{L}$, the boundary value can be used as the dominant factor in a perturbation expansion for $\mathcal{S}(\varpi)$ for $\varpi \in \mathcal{L}$. By rewriting (7.1), we arrive at an expression for a preconditioned operator:

$$\begin{aligned} A_{1/\varpi}^{-1} &= A_{1/\varpi_\infty}^{-1} + \mathcal{O}(\|\varpi - \varpi_\infty\|_1)\,, \\ A_{1/\varpi_\infty}^{-1} A_{1/\varpi} &= I \qquad\;\; + \mathcal{O}(\|\varpi - \varpi_\infty\|_1)\,. \end{aligned}$$

In our companion article [2], for preconditioning purposes, we study the boundary behavior of $\mathcal{S}$ for $n > 1$. We prove the strong convergence of $\mathcal{S}(\varpi_\nu)$ for any monotonically decreasing $(\varpi_\nu)_{\nu\in\mathbb{N}}$. In [2], we also characterize the associated limits for particular cases. Furthermore, a characterization of the limit of the discretized inverse operators with piecewise constant coefficients is given by the first author in [4] using linear finite element and finite volume methods. Furthermore, a similar characterization associated to the biharmonic plate equation is also given [5]. As expected, the limits are structurally simpler. Therefore, utilizing the limits as preconditioners should lead to computationally feasible preconditioning. One such approach was taken by the first author in [3, 4, 5] with rigorous justification using singular perturbation analysis. In [3], the proposed preconditioners were shown to be as effective as the algebraic multigrid preconditioner with an equivalent computational complexity. However, rigorous justification for algebraic multigrid preconditioners is still lacking. For future research, for instance, showing the effectiveness of a proposed preconditioner, we plan to utilize operator theory to prove spectral equivalences. For the derivation of such equivalences, operator theory provides the natural framework. Hence, the tools utilized in this article can be seen as steps towards adding tools to the arsenal of methods of analysis for rigorous justification.

7.2. Additional remarks. It is unclear whether results similar to those of the previous section can be expected to hold in dimensions greater than 1. According to Theorem 5.3 and the subsequent example, and different from the situation in one dimension, $\hat{A}$ is not injective when ϖ vanishes on nonempty open subsets of the material. Hence there does not seem to be an obvious candidate for a limit of a sequence of $\hat{A}^{-1}$ that is associated to a sequence in $\mathcal{L}$ approaching such ϖ. Therefore, it is conceivable that such limits show a wider variety of phenomena than those in one dimension. This problem deserves further study.

Alternative to the approach of the paper, in the definition of the operator A_p, it is possible to define the divergence operator as a weak divergence operator. In this way, an alternative definition of A_p can be given that leads to a bounded linear and strictly positive self-adjoint operator $\tilde{A}_p$ on $H_0^1(\Omega)$. This definition is given in the appendix along with the connection of the inverses of A_p and $\tilde{A}_p$.

The connection of the inverses of A_p and $\tilde{A}_p$ suggests that results similar to that of this paper might also hold for the operators $\tilde{A}_p$, but further study is necessary for a conclusive result in this direction.

A final remark concerns the fact that it cannot be expected that general *elliptic regularity theorems* hold for operators A corresponding to discontinuous diffusivities p as a consequence of the condition that every element u from the domain of such an operator satisfies $p\,\nabla_w u \in D(\nabla_0^*)$. For instance, the source function f in Example 2 is in $H^k(I)$ for every $k \in \mathbb{N}$, but $u = A^{-1}f \notin H^2(I)$, where I is the open interval $(-1,1)$ of $\mathbb{R}$.

Appendix. The following gives an abstract version of an alternative definition of the operators A_p.

THEOREM A.1. *Let X be a nontrivial complex Hilbert space, and let $A : D(A) \to X$ be a densely defined, linear, self-adjoint, and strictly positive operator in X and $Y := D(A^{1/2})$. Further, let $\langle\,|\,\rangle_Y : Y^2 \to \mathbb{C}$ be a scalar product on Y such that $(Y, \langle\,|\,\rangle_Y)$ is a complex Hilbert space, the inclusion $\iota : (Y, \langle\,|\,\rangle_Y) \to X$ is continuous, and $A^{1/2} : (Y, \langle\,|\,\rangle_Y) \to X$ is continuous.*

(i) *For each $\xi \in X$, there is a unique element $\iota_2(\xi) \in Y$ such that*

$$\langle \xi | \cdot \rangle\,|_Y = \langle \iota_2(\xi) | \cdot \rangle_Y \ ,$$

and the induced map $\iota_2 : X \to Y$, that associates $\iota_2(\xi)$ to every $\xi \in X$, is continuous.

(ii) *There is a unique map $\tilde{A} : Y \to Y$ such that*

$$\langle A^{1/2}\xi | A^{1/2}\xi' \rangle = \langle \xi \,| \tilde{A}\xi' \rangle_Y$$

for all $\xi, \xi' \in Y$. In particular, this map $\tilde{A}$ defines a strictly positive self-adjoint bounded linear operator on Y and satisfies

$$\tilde{A}^{-1} \circ \iota_2 = A^{-1} \ .$$

Proof. Theorem A.1(i): First, since the inclusion $\iota : (Y, \langle\,|\,\rangle_Y) \to X$ is continuous, for each $\xi \in X$, $\langle \xi | \cdot \rangle \circ \iota \in L(Y, \mathbb{C})$. Hence according to the Riesz representation theorem, there is a unique $\iota_2(\xi) \in Y$ such that $\langle \iota_2(\xi) | \cdot \rangle_Y = \langle \xi | \cdot \rangle \circ \iota$. Hence ι_2 is well defined. Further, ι_2 is obviously linear. In addition, it follows for $\xi \in X$ that

$$\|\iota_2(\xi)\|_Y^2 = |\,\langle \xi | \iota_2(\xi) \rangle\,| \leqslant \|\xi\|\,\|\iota_2(\xi)\| = \|\xi\|\,\|\iota(\iota_2(\xi))\| \leqslant \|\iota\|\,\|\xi\|\,\|\iota_2(\xi)\|_Y \, ,$$

where $\|\ \|_Y$ denotes the norm on Y that is induced by $\langle\ |\ \rangle_Y$, and hence that $\|\iota_2(\xi)\|_Y \leqslant \|\iota\|\,\|\xi\|$. As a consequence, ι_2 is continuous and $\|\iota_2\| \leqslant \|\iota\|$.

Theorem A.1(ii): Since $A^{1/2} : (Y, \langle\,|\,\rangle_Y) \to X$ is continuous, there is $c \geqslant 0$ such that

$$\|A^{1/2}\xi\| \leqslant c\,\|\xi\|_Y$$

for every $\xi \in Y$. In addition, by

$$s(\xi, \xi') := \langle A^{1/2}\xi | A^{1/2}\xi' \rangle$$

for every $\xi, \xi' \in Y$, there is defined a Hermitian sesquilinear form $s : Y^2 \to \mathbb{C}$ which satisfies

$$|s(\xi, \xi')| \leqslant \|A^{1/2}\xi\|\|A^{1/2}\xi'\| \leqslant c^2\,\|\xi\|_Y\|\xi'\|_Y$$

for all $\xi, \xi' \in Y$. As a consequence, by the Riesz representation theorem, to every $\xi \in Y$, there corresponds a unique element $\tilde{A}\xi \in Y$ such that

$$s(\xi, \cdot) = \langle \tilde{A}\xi | \cdot \rangle_Y \ .$$

As a consequence of that uniqueness, by

$$\tilde{A} := (Y \to Y, \xi \mapsto \tilde{A}\xi)$$

there is given a linear map. Further,

$$\|\tilde{A}\xi\|_Y^2 = |s(\xi, \tilde{A}\xi)| \leqslant c^2 \|\xi\|_Y \|\tilde{A}\xi\|_Y \ ,$$

and hence

$$\|\tilde{A}\xi\|_Y \leqslant c^2 \|\xi\|_Y$$

for every $\xi \in Y$. The last inequality also implies that $\tilde{A} \in L(Y, Y)$. Since

$$\langle \xi | \tilde{A}\xi' \rangle_Y = \langle \tilde{A}\xi' | \xi \rangle_Y^* = (s(\xi', \xi))^* = s(\xi, \xi') = \langle \tilde{A}\xi | \xi' \rangle_Y$$

for all $\xi, \xi' \in Y$, it follows that $\tilde{A}$ is self-adjoint. Since $A^{1/2} : (Y, \langle\,|\,\rangle_Y) \to X$ is continuous and bijective, it follows that $A^{-1/2} := (A^{1/2})^{-1} \in L(X, Y)$, and hence the existence of $d > 0$ is such that

$$\|A^{-1/2}\eta'\|_Y \leqslant d\,\|\eta'\|$$

for all $\eta' \in X$. This implies that

$$\frac{1}{d}\,\|\xi\|_Y \leqslant \|A^{1/2}\xi\|,$$

and hence that

$$\langle \xi | \tilde{A}\xi \rangle_Y = s(\xi, \xi) = \|A^{1/2}\xi\|^2 \geqslant \frac{1}{d^2}\,\|\xi\|_Y^2$$

for every $\xi \in Y$. Hence $\tilde{A}$ is strictly positive. In particular, it follows for $\eta \in Y$ that

$$\langle A^{1/2}\tilde{A}^{-1}\iota_2(\eta) | A^{1/2}\xi' \rangle = s(\tilde{A}^{-1}\iota_2(\eta), \xi') = \langle \tilde{A}\tilde{A}^{-1}\iota_2(\eta)\,|\,\xi' \rangle_Y = \langle \iota_2(\eta)\,|\,\xi' \rangle_Y = \langle \eta | \xi' \rangle$$

for every $\xi' \in Y$, and hence that $\tilde{A}^{-1}\iota_2(\eta)$ is a weak solution of

$$A\xi = \eta \ .$$

Hence it follows that $A^{-1}\eta = \tilde{A}^{-1}\iota_2(\eta)$ and, therefore, that

$$\tilde{A}^{-1} \circ \iota_2|_Y = A^{-1}|_Y \ .$$

Since Y is dense in X and ι is continuous, the last inequality implies that

$$\tilde{A}^{-1} \circ \iota_2 = A^{-1} \ . \qquad \square$$

Remark 7. We note that, e.g., $\langle\,|\,\rangle_Y := \langle\,|\,\rangle_{A^{1/2}}$, where

$$\langle \xi | \eta \rangle_{A^{1/2}} := \langle \xi | \eta \rangle + \langle A^{1/2}\xi | A^{1/2}\eta \rangle$$

for all $\xi, \eta \in D(A^{1/2})$, satisfies the assumptions of Theorem A.1. Hence in the case $A = A_p$, $(Y, \langle\,|\,\rangle)$ can be chosen as $H_0^1(\Omega)$.

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