On the Role of Computation in Mathematics

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Abstract

We analyze the mathematical significance of computed solutions to differential equations from a perspective of well-posedness and residual-based a posteriori error estimation in the setting of the incompressible Euler equations.

1 The Clay Navier-Stokes Millennium Problem

The Clay Mathematics Institute Millennium Problem on the incompressible Navier-Stokes equations worth $1 million, asks for a proof of the global existence or blowup/breakdown of smooth solutions. Since the official formulation of the Problem [5] is given by a pure mathematician and not a computational mathematician, it may be expected that only an analytical proof would be accepted. In the case of global existence of smooth solutions for smooth data, it seems that only an analytical proof can cover all data, but in the case of breakdown for a specific choice of data, it is conceivable that a computational proof can be sufficient.

In this note we address the analogous problem for the incompressible Euler equations, which for some unknown reason is not a Millennium Problem, although mentioned briefly in [5]. We observe that initially smooth computational solutions of the Euler equations exhibit blowup to turbulent solutions in finite time. The question is if this evidence could be worth $1 million.

This leads into the general question of the significance of computational solutions to differential equations. The basic question is what quality requirements should be put on computed solutions of differential equations to make them comparable to analytical solutions. We shall see that an affirmative answer can be given in terms of *well-posedness* and *residual-based a posteriori error estimation*.

2 The Euler Blowup Problem

The Euler equations express conservation of mass, momentum and total energy of a fluid with vanishingly small viscosity (inviscid fluid). In the case of *compressible* flow, it is well known that initially smooth solutions to the Euler equations in general develop into discontinuous *shock solutions* in finite time. Such shock solutions thus exhibit *blowup* in the sense that they have infinitely large derivatives and *Euler residuals* at the shock violating the Euler equations pointwise. The formation of shocks shows *non-existence of pointwise solutions* to the compressible Euler equations. Concepts of *weak solution* have been developed accomodating after-blowup shock solutions with Euler residuals being large in a strong (pointwise) sense and vanishingly small in a weak sense, but both the existence and uniqueness of weak solutions represent open problems since long.

Regularized Euler equations are augmented by a viscous term with small positive viscosity coefficient with the effect that the blowup to infinity of derivates and Euler residuals is curbed. Existence of pointwise solutions to suitably regularized Euler equations, referred to as *viscosity solutions*, follows by standard analytical techniques, see [3, 10]. *Viscosity solutions* have pointwise large (and weakly small) Euler residuals as a reflection of the non-existence of pointwise solutions to the Euler equations.

Proving convergence of viscosity solutions to weak solution limits of the Euler equations under vanishing viscosity, has remained a main challenge to analytical mathematics since the 1950s, but the progress has been limited to model problems; the difficulty is related to the non-existence of pointwise solutions and lack of viscosity in the Euler equations. No progress on this problem is reported, and since well-posedness is not included, the interest is unclear. This indicates that the object of mathematical study should concern viscosity solutions (which do exist), and the basic problem is then *output uniqueness* of viscosity solutions under vanishing viscosity, that is, what aspects or *outputs* of viscosity solutions converge (are stable) as the viscosity tends to zero. The Euler residual of a viscosity solution converges weakly to zero and thus represents a (trivial) stable output, and the question is if there are other more informative stable outputs.

Incompressible flow does not form shocks and one may ask if initially smooth solutions of the incompressible Euler equations exhibit blowup or not? The existing literature, see [3, 6, 7, 8] and references therein is not decisive and evidence for both blowup and not blowup is presented. The study has further been limited in time to before blowup, discarding the highly relevant question of what happens after blowup.

The purpose of this note is to adress the problem of blowup for the incompressible Euler equations drawing from our recent work [9] and references therein, widening the study to both before and after blowup. The motivation comes from the statement in [4] that this is "a major open problem in PDE theory, of far greater physical importance than the blowup problem for Navier-Stokes equations, which of course is known to the nonspecialists because it is a Clay Millenium Problem". We will argue that the problem does not have this significance, and essentially is solved.

We show that the phenomenon of *turbulence* in incompressible flow, plays a similar role in blowup as that of shock formation in compressible flow: Initially smooth solutions of regularized incompressible Euler equations in general in finite time show blowup into *turbulent solutions*, characterized by pointwise large (weakly small) Euler residuals and substantial *turbulent dissipation*. We show that the blowup into turbulence results from pointwise instability, forcing smooth solutions to develop into turbulent solutions, as a parallel to the inevitable shock formation in compressible flow. Since viscosity solutions are turbulent with derivatives becoming unbounded in turbulent regions, a limit would there be nowhere differentiable and have infinitely small scales, like a very complex Weierstrass function. No evidence for the existence of such a limit seems to be available.

We summarize our evidence for blowup as follows:

(a) Initially smooth solutions to the Euler equations, for example stationary potential solutions, are linearly pointwise exponentially unstable. The exponential instability is in particular expressed by vortex stretching.

(b) Computed viscosity solutions develop in finite time pointwise large Euler residuals and substantial turbulent dissipation under mesh refinement/vanishing viscosity.

(c) If a smooth solution to the regularized Euler equations had existed under vanishing viscosity, then corresponding computed solutions would have had small Euler residuals under sufficient mesh refinement.

Here (a) and (c) represent analytically provable facts, while (b) follows by inspecting computed viscosity solutions obtained using a *least squares residul-stabilized finite element method* referred to as Euler General Galerkin or *EG2*.

Together, (b) and (c) give evidence that initially smooth solutions to the Euler equations show blowup in finite time under vanishing viscosity, which is also supported by (a). Since (b) is computational and simulates vanishing viscosity with mesh refinement, the evidence is not only analytical.

If now turbulent blowup is shown to be inevitable, it is natural to study turbulent flow after blowup, more precisely what aspects of turbulent solutions are stable under vanishing viscosity. We shall see that global mean values such as drag and lift are stable, which we refer to as *weak output uniqueness*, and motivate as an effect of cancellation in a dual linearized problem.

3 Significance of Computed Solutions

Since our evidence partly is computational, we start with some general remarks on the mathematical significance of computed solutions to differential equations such as the Euler and Navier-Stokes equations.

First, suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with derivative f', and assume that for some $X \in \mathbb{R}$,

$$S|f(X)| < TOL, \quad (1)$$

where $S = |f'(X)^{-1}|$, and $|f'(y) - f'(X)| < \frac{1}{2}$ for $|y - X| < TOL$ with TOL a (small) positive error tolerance. Then there is $x \in \mathbb{R}$ with $|x - X| < TOL$ such that $f(x) = 0$. This follows from the contraction mapping principle applied to the mapping $x \to x - f'(X)^{-1}f(x)$, and can be generalized to Banach spaces.

Thus, the existence of a computed approximate root X with sufficiently small *residual* $R = f(X)$ modulo the stability factor S, guarantees the existence of an exact root x with vanishing residual $f(x) = 0$ within a distance of SR. A result of this nature for the Navier-Stokes equations is given in [2].

Since Hadamard it is well understood that solving differential equations, perturbations of data have to be taken into account. If a vanishingly small perturbation can have a major effect on the solution, then the problem is not *well-posed* and potentially is meaningless from both mathematical and applications points of view. In a well-posed problem, small perturbations have small effects on the solution. Perturbations of the equation $f(x) = 0$ can have the form $f(x) = R$ with a small R. An approximate root X of $f(x) = 0$ with residual $R = f(X)$, thus can be viewed as an exact root of the perturbed equation $f(x) = R$. It is thus natural to view both an exact root x and an approximate root X as representatives of a family of roots to perturbed equations. In this sense the approximate root X as as good a representative as an exact root x. The advantage of a computed solution is that is available for inspection, while an exact root only known to exist (e.g. by contraction mapping) in general cannot be inspected. For example, the computation $2 - 1.4^2 = 0.04$, shows that $\sqrt{2} \approx 1.4$ with an error less than $(2.8)^{-1}0.04 < 0.015$, while inspecting the mere symbol $\sqrt{2}$ reveals nothing about its value.

To compute a solution requires a specific choice of data (a specific function f) and thus it is impossible to compute the solutions for all data (for all functions f). However, it may well be possible to compute solutions for a finite set of data of relevance, including $f(x) = 2 - x^2$.

EG2 solutions are specific viscosity solutions of the Euler equations with specific viscosity arising from least squares penalization of the Euler residual, roughly speaking corresponding to introducing a standard viscosity proportional to the mesh size. Thus an EG2 solution is as much a viscosity solution as any exact viscosity solution, with the immense advantage that it is available for inspection, whereas a viscosity solution only proved to exist analytically, is not. Inspecting EG2 solutions we find that they are turbulent, in the sense that their Euler residuals are small weakly but large strongly, and that certain mean-value outputs are stable under mesh refinement/vanishing viscosity.

The above discussion is motivated by a common attitude among both pure and applied mathematicians that computed solutions to differential equations "prove nothing". This belief is probably rooted in the classical approach to the error analysis of difference methods based on the concept of *truncation error* obtained inserting the exact solution into the difference equation. However, this corresponds to a thought experiment since the exact solution is unknown along with the truncation error. In the a posteriori error analysis of finite element methods, instead the computed solution is inserted into the differential equation resulting in the residual which thus is known. In the classical error analysis of finite difference methods the unknown exact solution is given the major role, while in the a posteriori error analysis of finite element methods, the known computed solution has the entire role and is much a solution as any exact solution to the unperturbed equations.

Of course, computed solutions can only cover a finite sample of data, but the sample may include the data of interest. Morover, EG2 solutions come with *quantitative a posteriori output error control*, which is a new feature not available in classical computation often viewed as some form of magic without mathematical value. But EG2 solutions are not magical; they are (simply) specific viscosity solutions with output error control.

4 Smooth Solutions and Non-Blowup

In [2] a technique related to (1) is presented for proving existence of a strong solution to the Navier-Stokes equations by computation: It is proved that (i) if an approximate solution has a sufficiently small residual, then a strong (smooth) exact solution exits, and (ii) if a smooth solution exists, then there is an approximate solution with sufficiently small residual. Note that (ii) is the same as (c). The net result of [2] is thus that existence of a smooth solution, and thus non-blowup, can be verified by computation. In other words the implication $notB \rightarrow notA$ is proved in [2], where $A =$ blowup of exact solution, and $B =$ blowup of Galerkin approximate solutions. The trouble with this implication is that $notB$ is never true: Approximate solutions always show blowup under vanishing viscosity/mesh refinement. The implication $notB \rightarrow notA$ thus is empty.

On the other hand, the essence of our argument is the implication $B \rightarrow A$, which is applicable because we verify B to be true by computation. Note that $B \rightarrow A$ is logically equivalent to $notA \rightarrow notB$, that is: if the exact solution is smooth, then so is its Galerkin approximation, which is similar to (ii) and (c).

5 The Incompressible Euler Equations

We recall the Euler equations expressing conservation of momentum and mass of an incompressible inviscid fluid enclosed in a volume Ω in $\mathbb{R}^3$ with boundary Γ : Find the velocity $u = (u_1, u_2, u_3)$ and pressure p depending on $(x,t) \in \bar{\Omega} \times I$ with $\bar{\Omega} = \Omega \cup \Gamma$, such that

$$\begin{array}{rcll} \dot{u} + (u\cdot\nabla)u + \nabla p &=& f & \text{in } \Omega\times I, \\ \nabla\cdot u &=& 0 & \text{in } \Omega\times I, \\ u\cdot n &=& g & \text{on } \Gamma\times I, \\ u(\cdot,0) &=& u^0 & \text{in } \Omega, \end{array} \tag{2}$$

where n denotes the outward unit normal to Γ, f is a given volume force, g is a given inflow/outflow velocity, u^0 is a given initial condition, $\dot{u} = \frac{\partial u}{\partial t}$ and $I = [0,T]$ a given time interval. We notice the *slip boundary condition* expressing inflow/outflow with zero friction.

The Euler equations in the pointwise form (2) look deceptively simple, but have a major drawback: They cannot be solved in a pointwise sense, because of blowup by exponential instablity into turbulence!

6 Exponential Instability

The lack of viscosity with regularizing effect make the Euler equations "ill-posed" or "degenerate" and therefore unsuitable for mathematical study. This is indicated by formal linearization: Subtracting the Euler equations for two solutions (u,p) and $(\bar{u},\bar{p})$ with corresponding (slightly) different data, we obtain the following linearized equation for the difference $(v,q) \equiv (u-\bar{u}, p-\bar{p})$:

$$
\begin{aligned}
\dot{v} + (u\cdot\nabla)v + (v\cdot\nabla)\bar{u} + \nabla q &= f - \bar{f} && \text{in } \Omega\times I,\\
\nabla\cdot v &= 0 && \text{in } \Omega\times I,\\
v\cdot n &= g - \bar{g} && \text{on } \Gamma\times I,\\
v(\cdot,0) &= u^0 - \bar{u}^0 && \text{in } \Omega.
\end{aligned}
\tag{3}
$$

Formally, with u and $\bar{u}$ given, this is a linear convection-reaction problem for (v,q) with the reaction term given by the 3×3 matrix $\nabla\bar{u}$. By the incompressiblity, the trace of $\nabla\bar{u}$ is zero, which shows that in general $\nabla\bar{u}$ has eigenvalues with real value of both signs, of the size of $|\nabla u|$ (with $|\cdot|$ som matrix norm), thus with at least one exponentially unstable eigenvalue. We conclude that the perturbations in data in general shows local exponential growth with exponent $|\nabla u|$ and thus any solution (u,p) of the Euler equations (with non-vanishing ∇u), is locally exponentially unstable with exponent $|\nabla u|$. In particular, a stationary potential solution is exponentially unstable, because of the inevitable presence of small perturbations. The effect of the exponential instability is the development of turbulence with $|\nabla u|$ large (with very strong local exponential instability), as an analogue to the formation of shocks in compressible flow.

Note in particular that perturbations of all data have to be taken into account, also in the forcing f, even if $f=0$. In particular, a computational solution can be viewed as a pointwise solution with perturbed forcing f corresponding to the Euler residual (assuming here for simplicity zero perturbation of incompressibilty).

7 The Vorticity Equation

Formally applying the curl operator $\nabla\times$ to the momentum equation we obtain the vorticity equation

$$\dot{\omega} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \nabla\times f \quad \text{in } \Omega, \tag{4}$$

which formally is a convection-reaction equation in the vorticity $\omega = \nabla\times u$ with coefficients depending on u, of the same form as the linearized equation (3), with similar properties of exponential instability referred to as *vortex stretching*. The vorticity is thus locally subject to exponential growth with exponent $|\nabla u|$.

In classical analysis it is often argued that from the vorticity equation (4), it follows that vorticity cannot be generated starting from potential flow with zero vorticity and $f=0$, which is *Kelvin's theorem*. But this is an incorrect conclusion, since perturbations of $\bar{f}$ of f with $\nabla\times\bar{f}\neq 0$ must be taken into account. What you effectively see in computations is local exponential growth of vorticity in vortex stretching, even if $\nabla\times f=0$, which is a main route of instability to turbulence.

8 Viscous Regularization

We define the *Euler residual* by

$$R(u,p) \equiv \dot{u} + (u \cdot \nabla)u + \nabla p - f, \tag{5}$$

which is the residual of the momentum equation, assuming for simplicity that the incompressiblity equation $\nabla \cdot u = 0$ is not subject to perturbations. *The regularized Euler equations* take the form: Find (u_ν, p_ν) such that

$$\begin{aligned} R(u_\nu, p_\nu) &= -\nabla \cdot (\nu \nabla u_\nu) && \text{in } \Omega \times I, \\ \nabla \cdot u_\nu &= 0 && \text{in } \Omega \times I, \\ u_\nu \cdot n &= g && \text{on } \Gamma \times I, \\ u_\nu(\cdot, 0) &= u^0 && \text{in } \Omega, \end{aligned} \tag{6}$$

where $\nu > 0$ is a small *viscosity*. Notice that we keep the slip boundary condition $u_\nu \cdot n = g$, which eliminates viscous boundary layers. The turbulence we will discover thus does not emanate from viscous boundary layers (which is a common misconception). We consider here for simplicity a standard ad hoc regularization and return to computational regularization below. Existence of a pointwise solution (u_ν, p_ν) of (6) (allowing ν to have a certain dependence on $|\nabla u|$), follows by standard techniques, see e.g. [3]. Notice that the Euler residual $R(u_\nu, p_\nu)$ equals the viscous term $-\nabla \cdot (\nu \nabla u_\nu)$, which suggests an interpretation of the viscous term in the form of the Euler residual.

The standard *energy estimate* for (6) is obtained by multiplying the momentum equation with u_ν and integrating in space and time, to get in the case $f = 0$ and $g = 0$,

$$\int_0^t \int_\Omega R(u_\nu, p_\nu) \cdot u_\nu \, dxdt = D(u_\nu; t) \equiv \int_0^t \int_\Omega \nu |\nabla u_\nu(s,x)|^2 dxds, \tag{7}$$

from which follows by standard manipulations of the left hand side,

$$K(u_\nu(t)) + D(u_\nu; t) = K(u^0), \quad t > 0, \tag{8}$$

where

$$K(u_\nu(t)) = \frac{1}{2} \int_\Omega |u_\nu(t,x)|^2 dx.$$

This estimate shows a balance of the *kinetic energy* $K(u_\nu(t))$ and the *viscous dissipation* $D(u_\nu; t)$, with any loss in kinetic energy appearing as viscous dissipation, and vice versa. In particular $D(u_\nu; t) \leq K(0)$ and thus the viscous dissipation is bounded (if $f = 0$ and $g = 0$). On the other hand, *turbulent solutions* of (6) are characterized by the fact that (for t bounded away from zero),

$$D(t) \equiv \liminf_{\nu \to \infty} D(u_\nu; t) >> 0 \tag{9}$$

expressing that the *turbulent dissipation is substantial* (bounded away from zero) as ν tends to zero. This is *Kolmogorov's conjecture*. which is supported by massive computational evidence of turbulent solutions satisfying pointwise

$$|\nabla u_\nu| \sim \frac{1}{\sqrt{\nu}}, \quad |R(u_\nu, p_\nu)| \sim \frac{1}{\sqrt{\nu}}. \tag{10}$$

Since $R(u_\nu, p_\nu) = -\nabla \cdot (\nu \nabla u_\nu)$, substantial dissipation with $D(u_\nu, t) >> 0$ reflects that the residual $R(u_\nu, p_\nu)$ is large pointwise. On the other hand, it follows by standard arguments from (8) that

$$\|R(u_\nu, p_\nu)\|_{-1} \leq \sqrt{\nu}, \tag{11}$$

where $\| \cdot \|_{-1}$ is the norm in $L_2(I; H^{-1}(\Omega))$. Thus there is strong evidence that $R(u_\nu, p_\nu)$ is large pointwise and small weakly, as announced. We now turn to the main issue of uniqueness of regularized turbulent solutions.

9 Weak Output Uniqueness

Hadamard showed that questions about uniqueness must be coupled to *stability* or *well-posedness*. Uniqueness without *quantitative stability* can be meaningless (although not acknowledged in the formulation of the Clay Millenium Problem on Navier-Stokes equations). We introduce a concept of *weak output uniqueness* measuring the difference in a certain (scalar) *output* $M(\cdot)$ of two regularized Euler solutions (u, p) and $(\bar{u}, \bar{p})$ in terms of the difference in Euler residual $R(\cdot)$, of the form

$$|M(u,p) - M(\bar{u},\bar{p})| \leq S\|R(u,p) - R(\bar{u},\bar{p}))\|_{-1} \leq S(\|R(u,p)\|_{-1} + \|R(\bar{u},\bar{p})\|_{-1}), \quad (12)$$

where $S = S(u, \bar{u}, M)$ is a *stability factor*. Typically, the output $M(\cdot)$ is a *mean-value* in space-time such as *drag* or *lift* (in bluff body flow). The stability factor S measures certain stability aspects of a *dual linearized problem* (similar to the linearized problem with reaction coefficient ∇u) with data related to the output, [9].

Since the Euler residuals of regularized solutions are weakly small, the variation in output $M(\cdot)$ can by (12) be small if S is not too large. The stability with respect to the output $M(\cdot)$ is is thus measured quantitatively in terms of S. If S is not too large ($S << \nu^{-1/2}$), then the output is stable.

A crude analysis using Gronwall type estimates indicates that the dual problem is pointwise exponentially unstable because the reaction coefficient is locally very large, in which case effectively $S = \infty$. This is consistent with massive observation that point-values of turbulent flow are non-unique or unstable.

The basic issue is then if mean-values of turbulent solutions may be unique, and if so why? We have studied this problem carefully in [9] analytically and computationally. We have shown for a variety of problems by computation that mean-values such as drag and lift, are unique, because the corresponding stability factors are of moderate size. We have thus given computational evidence that mean values such as drag and lift of a bluff body moving in a fluid with very small viscosity like air, is uniquely determined. We have motivated the favorable stability of the dual problem for mean-value outputs of turbulent solutions, as an effect of *cancellation* from the following two sources:

(i) rapidly oscillating reaction coefficients of turbulent solutions,

(ii) smooth data in the dual problem for mean-value outputs.

For a laminar solution there is no cancellation, and therefore not even mean-values are unique. This is d'Alembert's paradox: A potential laminar solution has zero drag, while an arbitrarily small perturbation will turn it into a turbulent solution with substantial drag. The drag of a laminar solution is thus non-unique in the sense that an arbitrarily small perturbation will change the drag substantially. The stability factor is infinite for a laminar solution because of lack of cancellation [9].

We can summarize our findings as follows: Since solutions to the incompressible Euler equations are pointwise exponentially unstable, they exhibit blowup into turbulent solutions with pointwise large but weakly small Euler residuals and substantial dissipation. Turbulent solutions show weak uniqueness, as a result of cancellation in the associated dual linearized problem with large oscillating reaction coefficient. Turbulent solutions thus arise as a result of instability, but the oscillating nature of turbulent solutions allow mean-value outputs to be unique. Thus, out of turbulent chaos, a form of mean-value order is resurrected by cancellation effects in the chaos. Of course this connects to statistics in the sense that mean-values are well determined and predictable while point-values are not. However, the setting is fully deterministic and no statistical data or analysis is required.

The turbulent dissipation can be seen as a penalty consuming kinetic energy because of a large pointwise Euler residual (turning large scale kinetic into heat energy as a form of small scale kinetic energy). The turbulent dissipation puts a limit to the size of the velocity gradients and vorticity and thus controls the blowup process allowing the flow to continue beyond blowup. Heuristically, the dissipation or penalty must be substantial in order to have an effect and prevent blowup to infinity. Or the other way around, if the turbulent dissipation is small, then it does not prevent blowup to infinity, which it has to do in order for the flow to continue. And the flow has to continue.

Shocks in inviscid compressible flow may be stable under regularization, in the sense that regularized solutions converge to a shock solution in a strong norm such as an L_1-norm. It is natural to ask if similarly there may be stable discontinuous, locally smooth, solutions as limits of solutions of regularized incompressible Euler equations? The answer seems to be no, because all solutions are exponentially unstable, and no case of convergence to a discontinuous locally smooth solution is known, in the incompressible case.

The phenomena of shock formation and turbulence share the qualities of pointwise large Euler residuals and substantial dissipation. We can thus view shocks/turbulence as a common feature always present in both compressible and incompressible slightly viscous flow, which causes blowup. We have here focussed on blowup in incompressible inviscid flow from turbulence, and consider blowup from turbulence in compressible flow in [10].

10 Regularization in Computation

Computing turbulent solutions to the Euler equations means solving regularized Euler equations with different forms of regularization depending on the computational method. We use *least squares stabilized finite element methods* referred to as *General Galerkin* or *G2*, where the viscosity arises from penalization of Euler residuals. In G2 the penalty directly connects to the violation, the presence of a non-zero Euler residual (as it should according the theory of criminology), and thus explains the occurence of viscous effects in Euler solutions in a new way, not simply as is usual assuming ad hoc that “there is always some small viscosity”. G2 explains that the viscosity arises from the impossibility to satisfy the momentum equation in a pointwise sense (resulting from the exponential instability). Intepretation of viscosity as a residual penalty seems to offer new aspects of the (mystical) phenomenon of viscosity, including the nature of shear and bulk viscosity [10].

A G2 solution (U, P) on a mesh with local mesh size $h(x, t)$ satisfies the following energy estimate (with $f = 0$ and $g = 0$)

$$K(U(t)) + D_h(U; t) = K(u^0), \tag{13}$$

where

$$D_h(U; t) = \int_0^t \int_\Omega hR(U, P)^2 \, dxdt, \tag{14}$$

is an analog of $D(u_\nu; t)$ with $h \sim \nu$. We see as announced that the G2 viscosity arises from penalization of a non-zero Euler residual. A turbulent solution is characterized by substantial dissipation $D_h(U; t)$ with $R(U, P) \sim h^{-1/2}$ pointwise, and

$$\|R(U, P)\|_{-1} \leq \sqrt{h}, \tag{15}$$

and the output M satisfies and analog of (12) of the form

$$|M(U, P) - M(\bar{U}, \bar{P})| \leq S(\|hR(U, P)\| + \|hR(\bar{U}, \bar{P})\|), \tag{16}$$

where $\|\cdot\|$ is e.g. the $L_2(\Omega \times I)$-norm. By the energy estimate we have

$$|M(U,P) - M(\bar{U},\bar{P})| \leq S\sqrt{h},$$

showing output uniqueness under mesh refinement if $S << h^{-1/2}$.

Relating the viscous dissipation to the Euler residual shows that the turbulent dissipation results from an impossibility of finding functions with small Euler residuals, which can be expected to be independent of the regularization. We see this phenomenon in shocks of compressible flow, where the the shock jump is stable under vanishing viscosity (while the width of the shock decreases to zero). We now present one computational example showing that mean-value outputs of incompressible flow such as drag and lift, are stable under vanishing viscosity/mesh refinement. A variety of similar results are available on the homepage of [9].

11 Flow around Circular Cylinder

We consider potential flow (stationary inviscid incompressible irrotational flow) of a fluid of unit density filling $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ and moving with velocity $(1, 0, 0)$ at infinity, around a circular cylinder of unit radius with axis along the x_3-axis. We recall that the potential flow is constant in the x_3-direction and is given (in polar coordinates (r, θ) in a (x_1, x_2)-plane) by the potential function

$$\varphi(r,\theta) = (r + \frac{1}{r})\cos(\theta)$$

with corresponding velocity components

$$u_r \equiv \frac{\partial \varphi}{\partial r} = (1 - \frac{1}{r^2})\cos(\theta), \quad u_s \equiv \frac{\partial u}{\partial s} \equiv \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -(1 + \frac{1}{r^2})\sin(\theta)$$

with streamlines given as the level lines of the conjugate potential function

$$\psi \equiv (r - \frac{1}{r})\sin(\theta).$$

We see from the streamline plot in the x_1x_2-plane in Fig 1 that the flow separates from the cylinder at the rear stagnation point C, from where the streamline given by $\theta = 0$ emanates. In $\mathbb{R}^3$ there is thus a plane of separation attached to the rear of the cylinder. The potential flow is exponentially unstable at the rear stagnation point C, where strong streamwise vorticity is generated through the vorticity equation

$$\dot{\omega}_1 + (u \cdot \nabla)\omega_1 = \frac{\partial u_1}{\partial x_1}\omega_1$$

since at the the rear stagnation/separation $\frac{\partial u_1}{\partial x_1} > 0$. The exponential instability of potential flow at its rear separation point, generates streaks of low pressure streamwise vorticity (like a vortex down the bathtub drain), which give rise to substantial drag, see Fig. 2, 3, 4 and 5. We see that the turbulent solution, which we refer to as Euler flow, is similar to potential flow in front and on top/bottom of the cylinder, but the appearance of turbulent streamwise vorticity giving rise to the drag, is a new feature, not present in the fictional zero-drag potential flow solution. The streamwise vortices arrange themselves in a zig-zag pattern of co-rotating tubes allowing alternating streams of fluid push beyond the stagnation plane $x_2 = 0$, as seen in the figures.

12 Summary

According to Hadamard, only (suitably) well-posed mathematical models can have any mathematical and physical significance. For well-posed problems computed solutions with (suitably) small residuals have no less mathematical significance than exact solutions, since computed solutions are exact solutions to perturbed problems, and perturbations must be taken into account. The advantage of a computed solution is that it is available for inspection, while an exact solution proved to exist by some anaytical argument such as a fixed-point theorem, in general is not available for inspection. We have proved blowup of solutions to the Euler equations by observing blowup of computed solutions and well-posed of mean-value outputs.

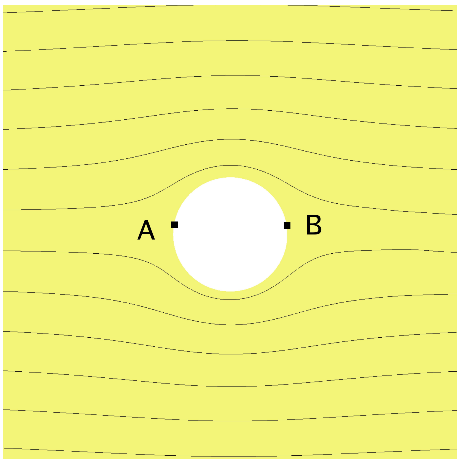

Figure 1: Potential fbw around a cylinder

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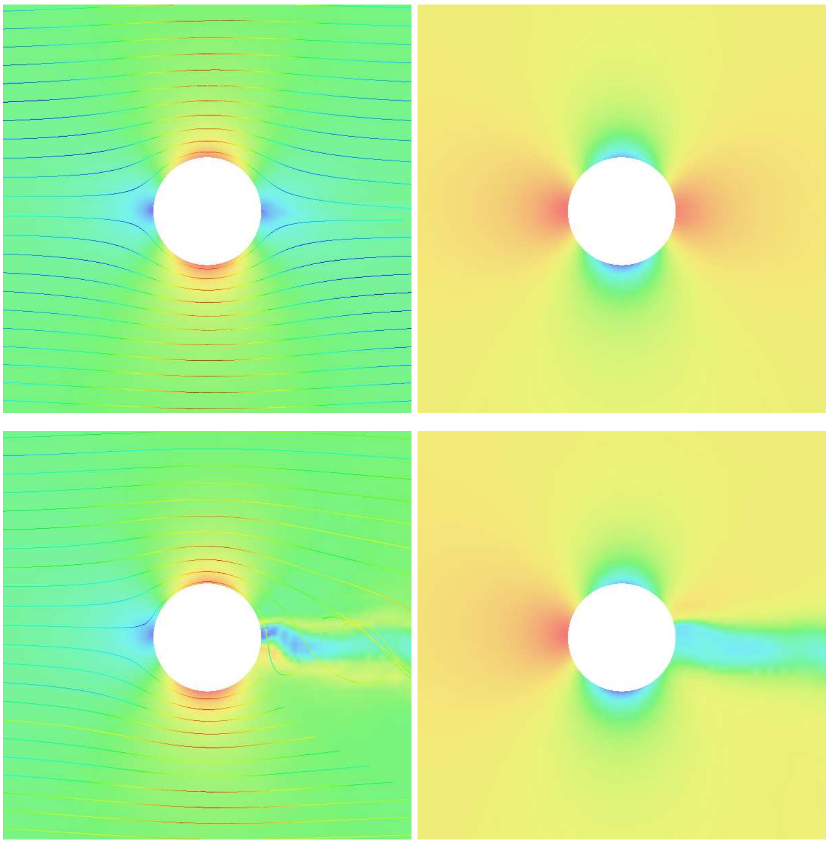

Figure 2: Flow past a circular cylinder; velocity (left) and pressure (right), for the potential solution (upper) and a G2 turbulent solution (lower).

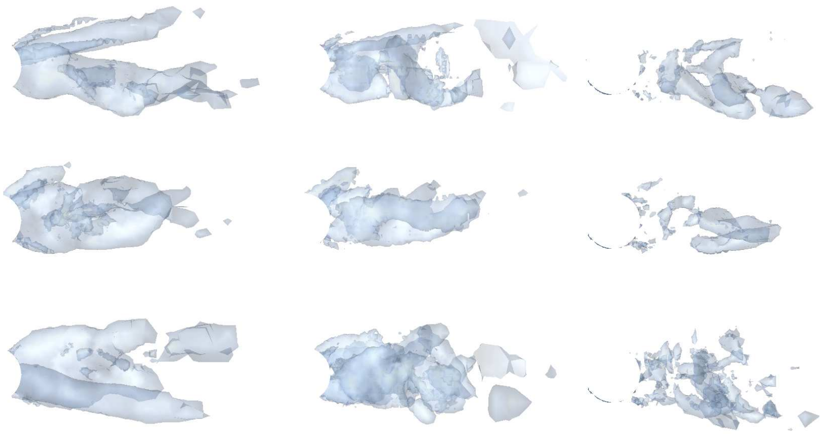

Figure 3: G2 simulation using a mesh with mesh size h; iso-surfaces for strong vorticity ($\sim h^{-1/2}$): $|\omega_1|$ (left), $|\omega_2|$ (middle) and $|\omega_3|$ (right), at three times $t_1 < t_2 < t_3$ (upper, middle, lower), in the x_1x_2-plane.

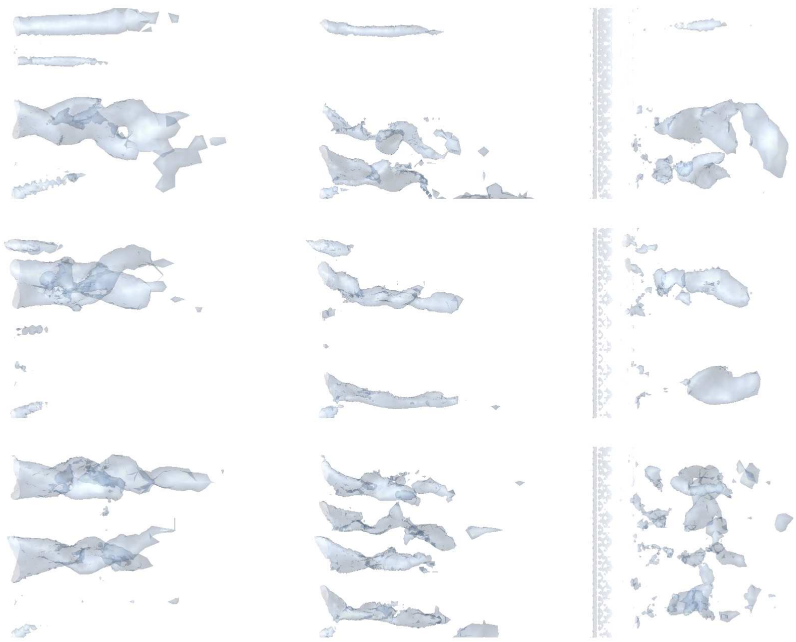

Figure 4: G2 simulation using a mesh with mesh size h; iso-surfaces for strong vorticity ($\sim h^{-1/2}$): $|\omega_1|$ (left), $|\omega_2|$ (middle) and $|\omega_3|$ (right), at three times $t_1 < t_2 < t_3$ (upper, middle, lower), in the x_1x_3-plane.

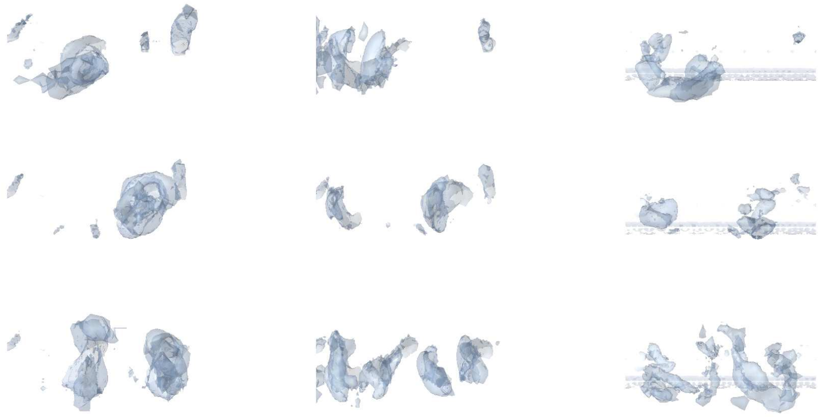

Figure 5: G2 simulation using a mesh with mesh size h; iso-surfaces for strong vorticity ($\sim h^{-1/2}$): $|\omega_1|$ (left), $|\omega_2|$ (middle) and $|\omega_3|$ (right), at three times $t_1 < t_2 < t_3$ (upper, middle, lower), in the x_2x_3-plane.

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